# Parameters Estimation via Monte Carlo for a GARCH Stochastic Volatility Model

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We introduced a variance reduction technique for Monte Carlo simulation in a GARCH stochastic volatility environment. Using least-squares and the accelerated Monte Carlo, we estimated parameters in GARCH model from the option prices observed in the market. The estimation results showed a mean-reversion pattern of the risk-free spot rate used in the model. This led us to consider Cox-Ingersoll-Ross (CIR) model for the spot rate and estimate its parameters. The numerical results were based on S&P500 option trade data.

*Key words:* Importance sampling for diffusions, GARCH model for volatility, CIR model for spot rate, parameters estimation.

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# 1 Introduction

Stochastic volatility models are widely used as alternatives to the Black-Scholes model in pricing of equity derivatives. In many cases, closed-form solutions were derived for the prices of European calls and puts. However, except in a few cases, no analytical solutions to the option pricing problem were derived with more general assumptions, e.g. when the spot rate is stochastic or when the option payoff involves several assets. Option pricing via Monte Carlo simulation is the only tool available in such cases. But even this path is not simple. Usually, the simulations involve two types of error. The first error arises from the discretization of the underlying process. The second one arises from Monte Carlo simulation of the discretized process. Normally, we can do little to reduce the error of the first type effectively. However, the error of the second type can often be reduced significantly.

The problem of reducing the error caused by the Monte Carlo simulation is directly connected with the problem of reducing the variance of the estimator of the option price. One of the methods that allow to reduce the variance is the importance sampling method, which was introduced for diffusion processes in [8]. However, an effective application of this method requires some specific knowledge of the process that is to be simulated. For example, in [1] and [2] an application of the importance sampling method was introduced for a class of stochastic volatility models. There, some approximations to the option prices in a stochastic volatility environment were used.

In this article, we derive a variance reduction technique for the GARCH stochastic volatility model with delay (see [5]). As an application of the technique, we present parameters estimation from the option prices traded in the market. We use least-squares optimization to find parameters of the model implied in the market prices. Its multiple iterations can now be afforded with a small computational cost. This is mainly because of the increased effectiveness of the Monte Carlo simulations for pricing options. In our results the GARCH stochastic volatility model shows a very good fit with the S&P500 index data and S&P500 option trade data.

The structure of this work is as follows. In Section 2 we briefly discuss the GARCH stochastic volatility model with delay. In Section 3 we review the option pricing approach for this model that was derived in [4] and [5]. In Section 4 we introduce the importance sampling for diffusions with delay. In Section 5 we consider an option price approximation that leads to the variance reduction for our GARCH model. In Section 6 we show the application of the variance reduction technique to the parameters estimation. We use the least-squares optimization to find the parameters from the S&P500 option trade data.

Several figures demonstrate our results.

## 2 The GARCH Stochastic Volatility Model

In [5] the following stochastic volatility model for stock price process $(S(t))_{0\le t\le T}$ was introduced:

$$\begin{aligned} dS(t) &= rS(t)\ dt + \sigma(t)S(t)\ dW(t), \\ \frac{d\sigma^2(t)}{dt} &= \gamma V + \frac{\alpha}{\tau}\left(\ln\frac{S(t)}{S(t-\tau)} - \mu\tau + \frac{1}{2}\int_{t-\tau}^{t}\sigma^2(u)\ du\right)^2 - (\alpha+\gamma)\sigma^2(t), \end{aligned} \tag{1}$$

where $W(t)$ is a standard one-dimensional Wiener process on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{0\le t\le T}, \mathcal{P})$, parameters $\alpha$, $\gamma$, $\tau$, $\mu$, $r$ and $V$ are positive. Here $r$ is a constant risk-free rate, $\mu$ is a drift rate of $S$ under a physical measure.

The main feature of this model is that it arises as a continuous-time analogue of GARCH(1,1) model

$$\begin{aligned} \ln\frac{S_n}{S_{n-1}} &\equiv u_n = \mu + \sigma_n\xi_n, \quad \{\xi_n\} \sim \text{i.i.d. N(0,1)} \\ \sigma_n^2 &= \gamma V + \alpha(u_{n-1}-\mu)^2 + (1-\alpha-\gamma)\sigma_{n-1}^2, \end{aligned}$$

and therefore its volatility possesses such important properties as mean reversion and clustering. Secondly, the equation for volatility does not contain another Wiener process as in most stochastic volatility models. However, the volatility $\sigma(t)$ in (1) is stochastic due to the presence of $S(t)$ and $S(t-\tau)$ in its drift coefficient. This feature ensures that the stock price model (1) constitutes a complete market. That is, any contingent claim on stock $S$ can be replicated in time by a trading strategy in shares of stock $S$ and a money account.

For the purpose of deriving an option pricing approach, we consider a simpler model for $S(t)$ similar to (1):

$$dS(t) = rS(t)\ dt + \sigma(t)S(t)\ dW(t), \tag{2}$$

where

$$\sigma^2(t) = \sigma_0^2 e^{-(\alpha+\gamma)t} + \left[\gamma V + \frac{\alpha}{\tau}\ln^2\left(\frac{S(t)}{S(t-\tau)}\right)\right]\frac{1-e^{-(\alpha+\gamma)t}}{\alpha+\gamma}.$$

Note that transition of the process $S$ at time $t$ is determined by its history over time interval $[t-\tau, t]$. Therefore, the initial data for (2) should be given on the interval $[-\tau, 0]$ and we denote it by $\varphi \in C([-\tau, 0], R)$. Parameter $\mu$ in model (2) was assumed 0 because of potential problems related to its estimation. Note however that this does not affect our model (1) since presumably $\mu$ takes very small values, e.g. 0.05.

# 3 Option Pricing

Now we review some facts on option pricing in the market where a stock price follows (2) (see [5]). The fair price of European option with terminal payoff $g(S(T))$ is given by the following conditional expectation

$$E\left[e^{-r(T-t)}g(S(T)) \mid \mathcal{F}_t\right]. \tag{3}$$

Using the Markov property for (2), the expectation is a functional of $S_t := \{S(u) : t-\tau \leq u \leq t\}$, which we denote $F(t, S_t)$.

We seek $F(t, S_t)$ in the following form

$$F(t, S_t) = \int_{-\tau}^{0} e^{-r\theta} H(S(t+\theta), S(t), t) d\theta, \tag{4}$$

where $H \in C^{0,2,1}(R \times R \times R_+)$. Then, $H(S(t+\theta), S(t), t)$ satisfies the following equation

$$0 = L(t, S_t) \equiv H|_{\theta=0} - e^{-r\theta} H|_{\theta=-\tau} + \int_{-\tau}^{0} e^{-r\theta} \left( H_3' + rS(t)H_2' + \frac{1}{2}\sigma^2(t, S_t)S^2(t)H_{22}'' \right) d\theta. \tag{5}$$

To find the option price $F(t, S_t)$, we can solve the equation subject to boundary condition

$$F(T, S_T) = g(S(T)).$$

A solution to equation (5) seems hard to find in a closed form. However, we can employ a finite difference scheme to solve the equation numerically (see [5]).

On the other hand, we can use Monte Carlo simulation of independent realizations $S^{(n)}(t)$ of the process $S(t)$ (discretized using the Euler scheme, see [6]) and approximate the expectation (3) with

$$F(t, S_t) \approx \frac{1}{N} \sum_{n=1}^{N} e^{-r(T-t)} g(S^{(n)}(T)). \tag{6}$$

Then, by the central limit theorem, the option price belongs to a confidence interval whose radius is proportional to the variance of the estimator. Normally, the variance is large and a substantial number of simulations are required to obtain a desired precision.

Both methods of finding the option price are more or less equivalent in terms of the computational efficiency. However, we can increase the efficiency in the second approach by reducing the variance of the estimator. With reduced variance, a smaller simulation time would be needed to get the desired precision of the estimator (6).

There is an efficient variance reduction technique, the so-called importance sampling for diffusions (see [8]). Employing the importance sampling variance reduction technique requires some approximation of the function $F(t, S_t)$. Using equation (5), we can approximate the option price and then use this approximation to derive a more efficient Monte Carlo estimator to (6).

# 4 Importance Sampling for Diffusions with Delay

In this section, we adapt a general formulation of the importance sampling technique introduced in [8]. Given a scalar square integrable $\mathcal{F}_t$-adapted process of the form $h(t, S_t)$, we define the following process

$$Q(t) = \exp\left\{\int_0^t h(u, S_u)\, du + \frac{1}{2}\int_0^t h^2(u, S_u)\, dW(u)\right\}.$$

If $E\left[Q(t)^{-1}\right] = 1$, then $(Q(t))_{0\le t\le T}$ is a positive martingale and we can define an equivalent to $\mathcal{P}$ probability measure $\mathcal{Q}$ through Radon-Nykodym density

$$\frac{d\mathcal{Q}}{d\mathcal{P}} = Q(T)^{-1}.$$

By Girsanov's theorem, the process defined by

$$W^{\mathcal{Q}}(t) = W(t) + \int_0^t h(u, S_u)\, du$$

is a standard Wiener process under the measure $\mathcal{Q}$. With respect to this new measure, the option price defined by $F(t, S_t)$ can be written

$$F(t, S_t) = E^{\mathcal{Q}}\left[e^{-r(T-t)} g(S(T)) Q(T) \mid \mathcal{F}_t\right].$$

We can estimate the expectation by

$$\frac{1}{N}\sum_{n=1}^{N} e^{-r(T-t)} g(S^{(n)}(T)) Q^{(n)}(T), \tag{7}$$

whose variance may be smaller than the variance of the estimator (6). Determining the function $h(t, S_t)$ that makes the variance smaller (or the smallest possible) is the sole goal of the *importance sampling method.*

The stock price process $S$ satisfies the following equation in terms of the new Wiener process $W^{\mathcal{Q}}$

$$dS(t) = [r - \sigma(t)h(t, S_t)]\, S(t)\ dt + \sigma(t)S(t)\ dW^{\mathcal{Q}}(t).$$

Using lemma 1 (see Appendix), we can write an equation for $F(t, S_t)$

$$dF(t, S_t) = [L(t, S_t) + rF(t, S_t)]\ \ dt + \sigma(t)S(t) \int_{-\tau}^{0} e^{-r\theta} H_2'(S(t+\theta), S(t), t) d\theta\ dW(t),$$

where operator $L(t, S_t)$ was defined in (5). Since $L(t, S_t) = 0$, we have

$$\begin{aligned} &d(F(t, S_t)Q(t)) = rF(t, S_t)Q(t)\ dt+ \\ &+ \left(F(t, S_t)h(t, S_t) + \sigma(t)S(t) \int_{-\tau}^{0} e^{-r\theta} H_2'(S(t+\theta), S(t), t)d\theta\right) Q(t)\ dW^{Q}(t), \end{aligned}$$

which leads to the following expression for variance

$$\begin{aligned} &\mathrm{Var}^{\mathcal{Q}}(g(S(T))Q(T)) = \\ &= E^{\mathcal{Q}} \int_0^T e^{2r(T-t)} Q^2(t) \left[F(t, S_t)h(t, S_t) + \sigma(t)S(t) \int_{-\tau}^{0} e^{-r\theta} H_2'(S(t+\theta), S(t), t)d\theta\right]^2 dt, \end{aligned}$$

as opposed to

$$\mathrm{Var}^{\mathcal{P}}(g(S(T))) = E^{\mathcal{P}} \int_0^T e^{2r(T-t)} \left[\sigma(t)S(t) \int_{-\tau}^{0} e^{-r\theta} H_2'(S(t+\theta), S(t), t)d\theta\right]^2 dt.$$

Therefore, the function $h(t, S_t)$ that makes the variance of the estimator (7) zero is

$$h(t, S_t) = -\frac{\sigma(t)S(t)}{F(t, S_t)} \int_{-\tau}^{0} e^{-r\theta} H_2'(S(t+\theta), S(t), t)d\theta. \tag{8}$$

Note that we could make the estimator of $F(t, S_t)$ "perfect" only if we knew the exact expression for $F(t, S_t)$. Fortunately, it is still possible to get a "good" estimator by approximating the function $F(t, S_t)$ and thus reducing the variance of the original estimator.

# 5 Option Price Approximation and Variance Reduction

Consider a European call option with maturity $T$ and payoff $g(S(T)) = \max(S(T) - K, 0)$. The fair price for this option at time $t$ when the stock price process follows GARCH model (2) is given by

$$F(t, S_t) = E\left[e^{-r(T-t)} \max(S(T) - K, 0) \mid \mathcal{F}_t\right].$$

This expectation can be estimated with (6), whose variance can be reduced using the importance sampling method. As it was mentioned in the previous section, we need to get an approximation to the option price in order to define a new measure that reduces the variance of estimator.

We approximate the option price $F(t, S_t)$ with the Black-Scholes call option price. Observe that the GARCH volatility (2) is mean reverting to the level $\sqrt{V}$. We can use this fact to approximate option price $F(t, S_t)$ with Black-Scholes price $F_{\text{BS}}(t, S(t))$, where constant volatility $\sqrt{V_{\text{BS}}}$ is used. Then similarly to (8), we choose function $h(t, S_t)$ as

$$h(t, S_t) = -\frac{\sigma(t)S(t)}{F_{\text{BS}}(t, S(t))} \frac{\partial F_{\text{BS}}(t, S(t))}{\partial S}.$$

On Figure 1 we show how the radius of 95% confidence interval of the option price estimator (7) varies as different values of $V_{\text{BS}}$ are chosen. Note that the minimum radius 0.0195 is reached at $V_{\text{BS}}$ close to the long-run variance rate $V = 0.0141$. The corresponding radius for the regular estimator (6) is 0.1328, and thus we reduced the radius of the confidence interval 6.8 times. This is equivalent to 46.4 times reduction of a number of realizations required to get a desired precision of the estimator.

The efficiency of the variance reduction can be judged from the following empirical viewpoint. If we were to simulate a Monte Carlo estimator of the call option price when the stock price volatility is constant, we could choose the estimator with theoretically zero variance since the option price in this case is known and it is given by the Black-Scholes formula. However, there is a discretization error in approximating the stock price process, and therefore the variance of the estimator is not zero anymore but close to it. We can use this lowest possible variance for a given time discretization step as a benchmark for the variance reduction in our model (2).

The radius of the confidence interval corresponding to the lowest possible variance in the constant volatility case is 0.0135. Since for our model (2) it is 0.0195, the efficiency of the variance reduction is very good. Moreover, this shows that using a relatively low-order

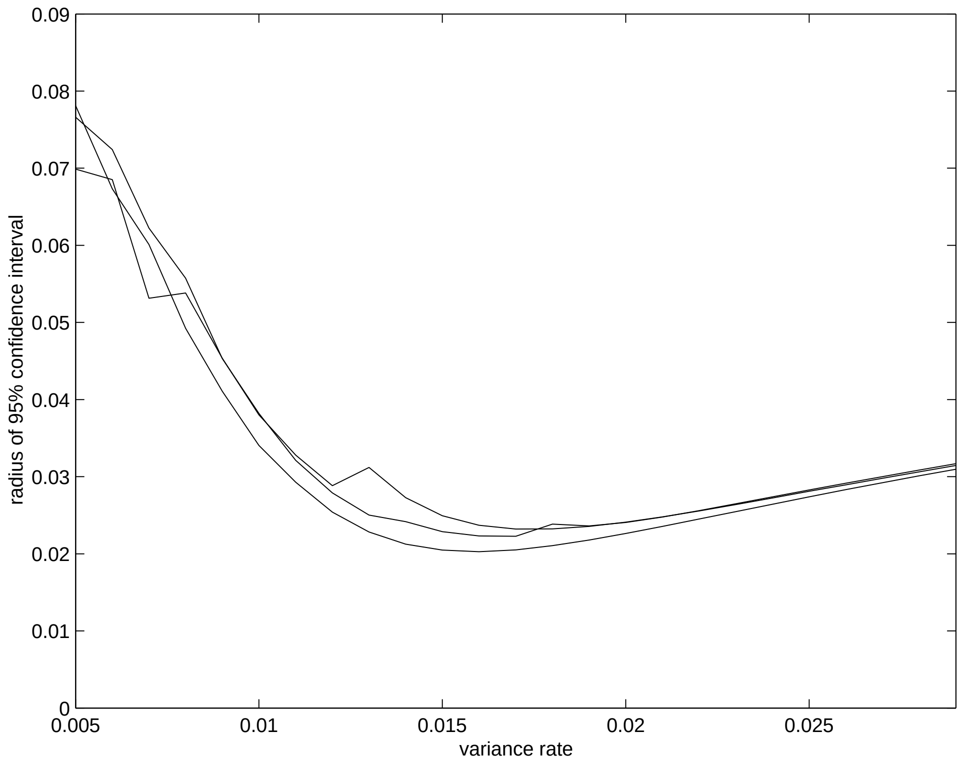


Figure 1: Radius of 95% confidence interval of the Monte Carlo estimator vs. variance rate $V_{\mathrm{BS}}$ used in the option price approximation. Several realizations are shown.

approximation of the option price we can get a significant variance reduction. This fact was observed in [2] and it is confirmed by our numerical results.

# 6 Parameters Estimation

Parameters of our stock price model (2) are

$$V = 0.0141, \quad \alpha = 0.0575, \quad \gamma = 0.0539, \quad \tau = 0.028, \quad \sigma_0^2 = 0.0111.$$

These parameters were estimated from S&P500 index data for years 1990-1993 (see [5]) using the maximum likelihood method. The only parameter that we could not estimate from the index data was the risk-free rate $r$. In this article, we use the S&P500 option trade data to imply this parameter.

Given the option price as a function $F$ of its unknown parameters $\theta$, we can find the values of the parameters that fit the option price data. The fit can be achieved using the following least-squares optimization

$$\min_{\theta} \sum_{i=1}^{n} (F(\theta, T_i, K_i) - C_i)^2,$$

where $F(\theta, T_i, K_i)$ is the European call option price as a function of unknown parameters $\theta$, time to maturity $T_i$ and strike price $K_i$; $C_i$ is the option price observed in the market.

The parameter $r$ is the only unknown parameter. For any fixed $r$, $T_i$ and $K_i$, the function $F(r, T_i, K_i)$ is given by the following expectation

$$F(r, T_i, K_i) = E\left[e^{-rT_i} \max(S(T_i) - K_i, 0)\right],$$

where $S(t)$ follows (2). It can be computed using the accelerated Monte Carlo from the previous section.

On Figure 2 we present numerical results on estimation of parameter $r$ from the market data. Observe that when the option maturity increases, the estimated risk-free rate decreases to some mean-reversion level. It is well-known that a bond spot rate is mean-reverting. Therefore, it is reasonable to assume that the risk-free rate implied in S&P500 options market possesses this property. Hence, it is of great interest to fit the rate $r$ with some of the well-known models of the spot rate, e.g. Cox-Ingersoll-Ross (CIR) model

$$dr(t) = a(m - r(t))\, dt + b\sqrt{r(t)}\, dW^r(t), \quad r(0) = r_0.$$

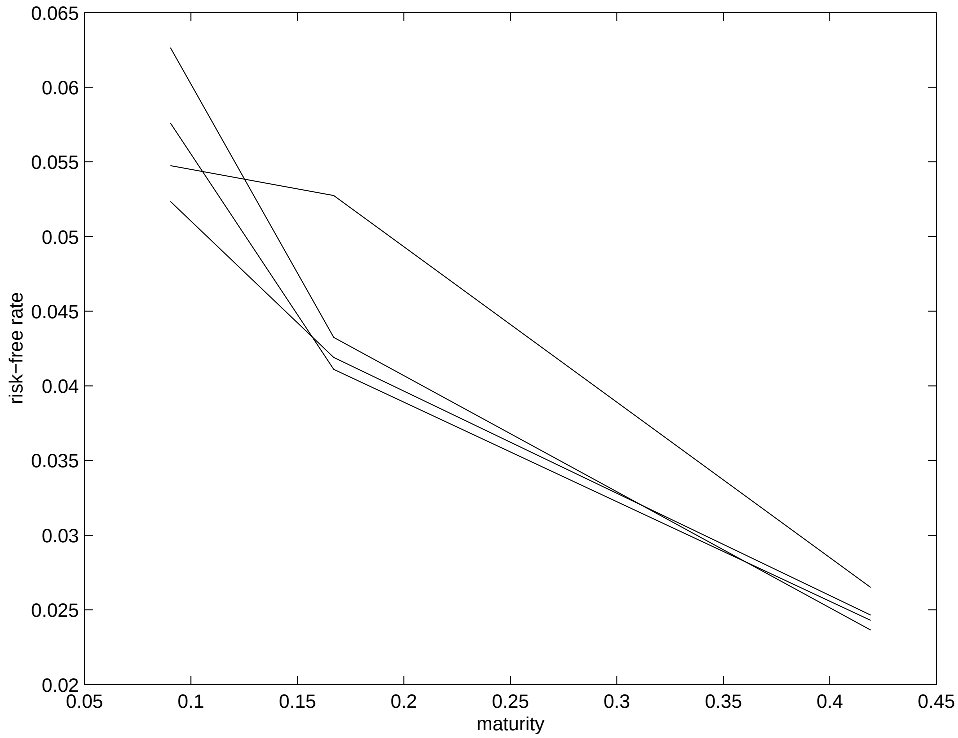


Figure 2: Constant risk-free rate $r$ implied from the market prices vs. maturity for different strike prices. The estimation is based on GARCH stochastic volatility model for S&P500 index.

In this case, the stock price process follows

$$dS(t) = r(t)S(t)\ dt + \sigma(t)S(t)\ dW(t)$$

with $\sigma(t)$ same as in (2), where $W^r(t)$ and $W(t)$ are assumed uncorrelated.

Estimation of parameters $m$, $a$, $b$ and $r_0$ is performed using the same least-squares approach. First, the European call option price is given by the following expectation

$$F(\theta, T_i, K_i) = E\left[\exp\left\{-\int_0^{T_i} r(u)du\right\}\max(S(T_i) - K_i, 0)\right],$$

where $\theta = (m, a, b, r_0)$. The corresponding option price estimator is

$$\frac{1}{N}\sum_{n=1}^{N}\exp\left\{-\int_0^{T_i} r^{(n)}(u)du\right\}\max(S^{(n)}(T_i) - K_i, 0).$$

Iteratively performing Monte Carlo simulations of the option price and comparing it with

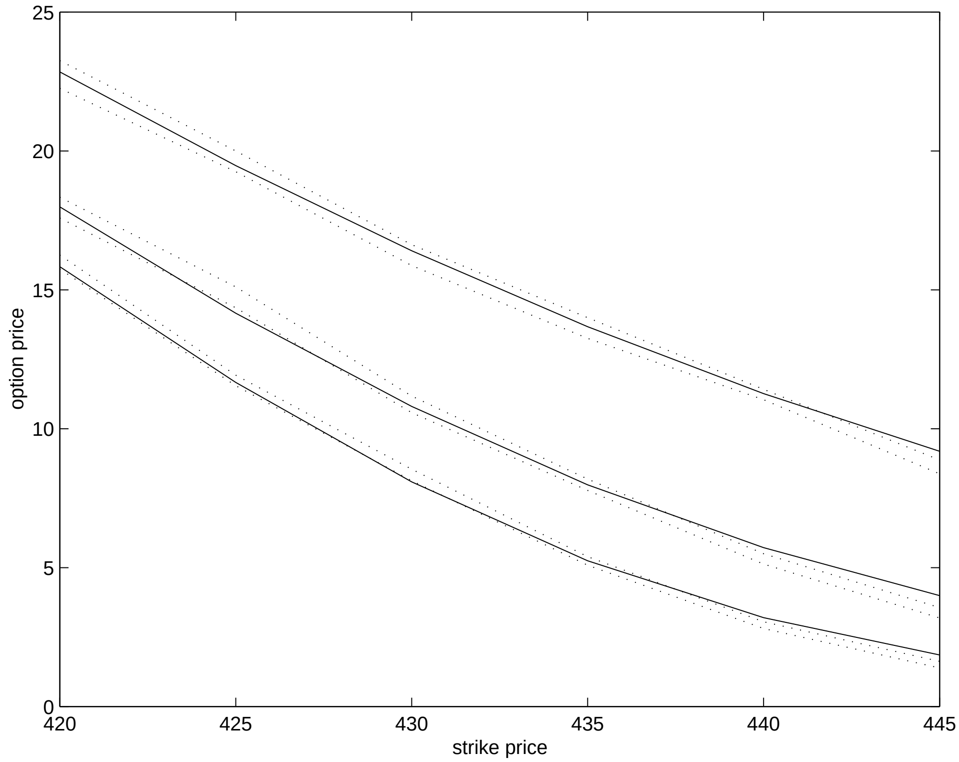


Figure 3: European call option price vs. strike price for three different maturities. Dashed lines represent bid and ask prices observed in the market and solid lines represent simulated prices based on CIR-GARCH stochastic volatility model.

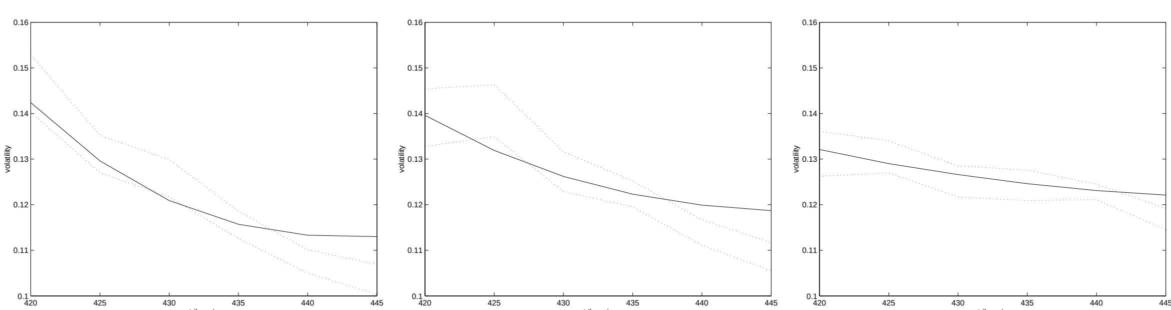

Figure 4: Implied volatility vs. strike price for three different maturities. The volatility plots correspond to the option prices depicted on Figure 3.

the market data, we obtained the following parameter estimates

$$m = 0.0075, \quad a = 8.5, \quad b = 0.3, \quad r_0 = 0.07.$$

On Figure 3 we show the fit of the simulated option prices with the prices observed in the market. Note that the fit is exact except for some out-of-money options. Figure 4 presents the corresponding implied volatility plots.

# Conclusion

In this article we developed a method to accelerate the Monte Carlo simulation used in option pricing for diffusions with delayed response. The method involved the variance reduction importance sampling technique adapted from [8]. The concrete diffusion model with delay used in option pricing was the GARCH continuous-time stochastic volatility model with delay introduced in [5].

The second purpose of the article was to develop estimation of the parameters involved in the model from the option trade data. Namely, the parameters used in the model for the risk-free spot rate. We assumed that the spot rate follows Cox-Ross-Ingersoll (CIR) model while the equity price follows already mentioned GARCH model. The estimation of parameters of the GARCH model from the index price data was previously derived in [5]. In order to estimate the parameters of the model for the spot rate, we had to use the option price data since the risk-free rate arises only in the risk-neutral evaluation of contingent claims, e.g. European call options. The estimation was carried out by the least-squares optimization where the accelerated Monte Carlo was iteratively used. Our results based on S&P500 data showed an exact fit of simulated option prices with the market prices except for some out-of-money options.

# Appendix

Here we formulate the following generalization of Ito's lemma, which was used in Section 4. The proof of this lemma is given in [4].

**Lemma 1** *Suppose a functional $F : R_+ \times C([-\tau, 0], R) \to R$ has the following form*

$$F(t, S_t) = \int_{-\tau}^{0} h(\theta) H(S(t+\theta), S(t), t) d\theta,$$

*for* $H \in C^{0,2,1}(R \times R \times R_+)$ *and* $h \in C^1([-\tau, 0], R)$. *Then*

$$F(t, S_t) = F(0, \varphi) + \int_0^t \mathcal{A}F(u, S_u)du + \int_0^t \sigma(u, S_u)S(u)\mathcal{B}F(u, S_u)dW(u), \tag{9}$$

*where for* $(t, x) \in R_+ \times C([-\tau, 0], R)$,

$$\begin{aligned}\mathcal{A}F(t, x) =& h(0)H(x(0), x(0), t) - h(-\tau)H(x(-\tau), x(0), t) - \\ & - \int_{-\tau}^0 h'(\theta)H(x(\theta), x(0), t)d\theta + \int_{-\tau}^0 h(\theta)LH(x(\theta), x(0), t)d\theta, \\ \mathcal{B}F(t, x) =& \int_{-\tau}^0 h(\theta)H_2'(x(\theta), x(0), t)d\theta,\end{aligned}$$

*and*

$$\begin{aligned}LH(x(\theta), x(0), t) =& rx(0)H_2'(x(\theta), x(0), t) + \frac{\sigma^2(t, x)x^2(0)}{2}H_{22}''(x(\theta), x(0), t) + \\ & + H_3'(x(\theta), x(0), t),\end{aligned}$$

*where* $H_i'$, $i = 1, 2, 3$, *represents the derivative of* $H(x(\theta), x(0), t)$ *with respect to i-th argument.*

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