Pricing of Derivatives in Security Markets with Delayed Response

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To My Family

Abstract

During the last three decades there has been a booming development in modeling of financial security markets, and a large number of models have been introduced following the pioneering work of Black and Scholes in 1973 on option pricing. One of the key assumptions in the Nobel-prize winning work of Black and Scholes is that the parameter of volatility is a constant. This controversial assumption generates a significant development of stochastic volatility models. Unfortunately, most stochastic volatility models constitute incomplete security markets, an inconvenient feature for the purpose of option pricing.

My study of security markets is aimed to develop a more general class of viable stochastic volatility models that constitute complete security markets. These models contain only one source of randomness, unlike the most stochastic volatility models studied in the literature. We allow a delayed response in the diffusion model for the price of the underlying asset, which may be a stock, an index or any other type of equity. The concept of delayed response is not new and it was introduced in the finance literature as a plausible explanation for abnormal behavior of equity returns. There is also statistical evidence in support of the past-dependence of equity returns.

The subject of this work is a theory of stochastic delay differential equations with applications to finance. In particular, we study stock price modeling and option pricing in the markets with delayed response. We introduce several continuous-time models for stochastic volatility with delay connected with the GARCH model, the widely used econometric model. In these models the time delay is considered constant. We illustrate the viability of the models by looking at the simulated implied volatility structure. We introduce a general option pricing approach for markets with delayed response. For those equations for which it is hard to find analytic solutions, we provide a numerical scheme. We also address the important issue of parameter estimation.

Based on stock market data, we show that there is a high variability of the estimated delay. In order to explain this, we extend the aforementioned models to include a state-dependent delay. As an essential tool, we derive a discrete-time approximation result for the so-called stochastic state-dependent delay differential equations. We show the influence of state-dependence of the delay on option pricing through a variety of U-shaped implied volatility plots.

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1 Introduction

In the early 1970's, Black and Scholes made a major breakthrough by deriving pricing formulas for vanilla options written on the stock (see [8]). Their model and its extensions assume that the probability distribution of the underlying cash flow at any given future time is lognormal. This assumption is not always satisfied by real-life options as the probability distribution of an equity has a fatter left tail and thinner right tail than the lognormal distribution (see [35]), and the assumption of constant volatility σ in financial models (such as the original Black-Scholes model) is incompatible with derivatives prices observed in the market.

1.1 Volatility modeling

Volatility modeling issues have been addressed and studied under different assumptions.

(i) Volatility is assumed to be a deterministic function of the time: $\sigma \equiv \sigma(t)$ (see [65]);

(ii) Volatility is assumed to be a function of the time and the current level of the stock price: $\sigma \equiv \sigma(t, S(t))$ (see [35]). The dynamics of the stock price satisfies the following stochastic differential equation:

$$dS(t) = \mu S(t)\, dt + \sigma(t, S(t))S(t)\, dW_1(t),$$

where $W_1(t)$ is a standard Wiener process;

(iii) The time variation of the volatility involves an additional source of randomness represented by $W_2(t)$ and is given by

$$d\sigma(t) = a(t, \sigma(t))\, dt + b(t, \sigma(t))\, dW_2(t),$$

where $W_2(t)$ and $W_1(t)$ (the initial Wiener process that governs the price process) may be correlated (see [12] and [36]);

(iv) The volatility depends on a random parameter: $\sigma \equiv \sigma(x(t))$, where $x(t)$ is some random process (see [21], [27], [61], [62], [63]).

In the approach (i), the volatility coefficient is independent of the current level of the underlying stochastic process $S(t)$. This is a deterministic volatility model, and the special case when σ is a constant reduces to the well-known Black-Scholes model that suggests changes in stock prices are lognormally distributed. But the empirical test in [9] seems to indicate otherwise. One explanation for this problem is the possibility that the variance of $\log(S(t)/S(t-1))$ changes randomly. This motivated the work [15], where the prices are analyzed for European options using the modified Black-Scholes model of foreign currency options and a random variance model. In their works the results from [36], [56] and [64] were used in order to incorporate randomly changing variance rates.

In the approach (ii), several methods have been developed to derive formula for a corresponding option price: one can obtain a formula by using stochastic calculus and, in particular, the Ito's lemma (see [54]). In [17], an alternative approach was developed, where a formula is interpreted as the continuous-time limit of a binomial random model. A generalized volatility coefficient of the form $\sigma(t, S(t))$ is said to be *level-dependent*. Because volatility and asset price are perfectly correlated, we have only one source of randomness given by $W_1(t)$. A time and level-dependent volatility coefficient makes the arithmetic more challenging and usually precludes the existence of a closed-form solution. However, the arbitrage argument based on portfolio replication and a completeness of the market remains unchanged.

The situation becomes different if the volatility is influenced by a second non-tradable source of randomness. This is addressed in the approaches (iii) and (iv) and one usually obtains a *stochastic volatility model*, which is general enough to include many deterministic models as special cases. The concept of stochastic volatility was introduced in [36], and subsequent development includes [64], [38], [56], [59] and [32]. We also refer to [24] for an excellent survey on level-dependent and stochastic volatility models.

There is yet another approach connected with stochastic volatility, namely, uncertain volatility scenario (see [12]). This approach is based on the uncertain volatility model developed in [4], where a concrete volatility surface is selected among a candidate set of volatility surfaces. This approach addresses the sensitivity question by computing an upper bound for the value of the portfolio under arbitrary candidate volatility, and this is achieved by choosing the local volatility $\sigma(t, S(t))$ among two extremal values $\sigma_{\min}$ and $\sigma_{\max}$ such that the value of the portfolio is maximized locally.

1.2 Modeling of delayed response

An assumption made implicitly by Black and Scholes in [8] is that the historical performance of the (B, S)-securities markets can be ignored. In particular, the so-called Efficient Market Hypothesis implies that all information available is already reflected in the present price of the stock and the past stock performance gives no information that can aid in predicting future performance. However, some statistical studies of stock prices (see [3] and [57]) indicate the dependence on past returns.

The issue of market's delayed response was raised by Bernard and Thomas in [7]. They analyzed the drift of estimated cumulative abnormal returns after earnings are announced. They observed that the returns continue to drift up for good news firms and down for bad news firms. They provided two possible explanations for this. Their first explanation suggests that at least a portion of the price response to new information is delayed. They explain that the delay might occur either because traders fail to assimilate available information, or because certain costs (such as transaction costs) exceed gains from immediate exploitation of information for a sufficiently large number of traders. A second explanation suggests that, because the capital-asset-pricing model used to calculate the abnormal returns is either incomplete or misestimated, researchers fail to adjust raw returns fully for risk. They came to the conclusion that their results are consistent with a delayed response to information. This is summarized in [6]: "The results of this paper cast serious doubt on any belief that asset pricing model misspecifications might explain post-earnings-announcement drift. An understanding of this anomaly appears to require either some model of inefficient markets, or identification of some cost (other than transactions costs) that impede the impounding of public information in prices". See [11], [28] and [40] for more evidence and analysis of past-dependence of stock returns.

There were some attempts to model the past-dependence. For example, in [43] it was obtained a diffusion approximation result for processes satisfying some equations with past-dependent coefficients, and this result was applied to a model of option pricing, in which the underlying asset price volatility depends on the past evolution to obtain a generalized (asymptotic) Black-Scholes formula. It was shown that the volatility is a deterministic function of time, which is determined by the initial stock price path. This implies that the option price is given by the Black-Scholes formula with some parameter of volatility. Therefore, the implied volatility plot for their model is flat with respect to the strike price.

A new class of non-constant volatility models was suggested in [33], which can be extended to include the aforementioned level-dependent model and share many characteristics with the stochastic volatility model. In the model suggested in [33], the past-dependence of the stock price process was introduced through volatility given as a function of exponentially weighted moments of historic log-price. This was done in such a way that the price and volatility formed a multi-dimensional Markov process. The model produced implied volatility skews of convex and concave shapes. The direction of the skew was determined by whether the asset price was below or above its recent average value.

In [37], the model of [33] was extended by analyzing the discrete-time model that is convergent to the continuous-time model in [33]. They showed that their model shares many common features with GARCH(1,1) model and that the pseudo maximum likelihood method can be applied to estimate the parameters involved.

Chang and Yoree [13] studied the pricing of a European contingent claim for the (B, S)-securities markets with a hereditary price structure. The price dynamics for the bank account and the evolution of the stock account are described by a linear functional differential equation and a linear stochastic functional differential equation, respectively. They showed that the rational price for a European contingent claim is given by an expectation of discounted final payoff, and that it is independent of the mean growth rate of the stock. Later in [14], they showed that Black-Scholes formula can be generalized to include the (B, S)-securities market with affine hereditary price structure.

Mohammed et al. [51] derived a delayed option price formula by solving a PDE similar to that of Black and Scholes. In their work, the volatility has a form $\sigma(S(t-b))$ for some delay parameter $b > 0$. When deriving the PDE, they assumed that the option price is a function of the time and the current value of the stock only.

In all papers mentioned above, authors showed that the past-dependence is an important feature of a stock price process and it should not be ignored. Therefore, it is imperative to introduce a general framework for its study, which is the main object of this thesis.

1.3 Structure of the thesis

The subject of this thesis is the study of Stochastic Delay Differential Equations (SDDE) that arise in modeling and option pricing for security markets with delayed response. The model for the equity price is the following SDDE

$$dS(t) = \mu S(t)\, dt + \sigma(t, S_t) S(t)\, dW(t),$$

where $S_t := \{S(t+\theta), \theta \in [-\tau, 0]\}$ and $\tau > 0$ is a time delay parameter.

The thesis is organized as follows. In Chapter 2, we show the existence of a risk-neutral probability measure and that the option price is given by the expectation of a discounted final payoff under this measure. In this Chapter we also derive a continuous-time equivalent of GARCH(1,1)-model for stochastic volatility with delay. We consider the GARCH model because it has been proved consistent with the stock market data and it is widely used in equity modeling.

In Chapter 3, we provide a general approach for pricing European call options written on a stock whose volatility is a continuous function of time and path S_t of the stock price process. By deriving an analogue of Ito's lemma, we obtain an integro-differential equation for a function of option price with boundary conditions specified according to the type of option to be priced. We solve this equation using a numerical scheme obtained through a finite-difference approximation of derivatives. We also provide an alternative way to price options through accelerated Monte Carlo simulations.

In Chapter 4, we address the issue of parameter estimation. We use the maximum likelihood method to estimate parameters (excluding time delay τ) that locally maximize the likelihood function. The delay parameter is chosen according to Akaike information criterion (AICC). This criterion is widely used in statistical inference for model selection. However, there is a parameter that cannot be estimated from the equity price data: the spot risk-free rate r that arises only in a risk-neutral evaluation. We estimate this parameter (or the yield curve) from market prices of options with different maturities. Our results show that the yield curve is not flat but can be fit with the Cox-Ingersoll-Ross (CIR) model. We estimate the parameters in CIR model using the least-squares method.

In Chapter 5, we extend the aforementioned models to include a varying time delay which depends on the values of the state, i.e. $S(t)$ and $\sigma(t)$. We prove the existence and uniqueness of the solution to a general stochastic state-dependent delay differential equation (SSDDE). We also prove the convergence of the Euler discrete-time approximation scheme for SSDDEs and establish the order of convergence. Using this approximation result, we perform Monte Carlo simulations of the stock price process with state-dependent delay and show viability of the model through a variety of implied volatility plots.

2 SDDEs in financial data modeling

In this Chapter, we introduce the general model of (B, S)-securities market with delayed response. For such a market, we show the existence of a risk-neutral probability measure and that the option price is given by the expectation of a discounted final payoff under this measure.

In the second part of this Chapter, we show that a model of (B, S)-securities market with delayed response arises as a continuous-time equivalent of GARCH(1,1)-model.

We start with the description of some known facts in the theory of stochastic delay differential equations (SDDEs).

2.1 Stochastic delay differential equations

For any path $x : [-\tau, \infty) \to R^d$ at each $t \geq 0$ define the segment $x_t : [-\tau, 0] \to R^d$ by

$$x_t(s) := x(t+s) \quad a.s., \ t \geq 0, \ s \in [-\tau, 0].$$

Denote by $C := C([-\tau, 0], R^d)$ the Banach space of all continuous paths $\eta : [-\tau, 0] \to R^d$ equipped with the supremum norm

$$\|\eta\| := \sup_{s \in [-\tau, 0]} |\eta(s)|, \quad \eta \in C,$$

where $|\cdot|$ is the Euclidean norm. Consider the following SDDE (see [50])

$$\begin{cases} dx(t) = H(t, x_t)\, dt + G(t, x_t)\, dW(t), \quad t \geq 0 \\ x_0 = \phi \in C \end{cases} \tag{2.1}$$

on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the following conditions: the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and each $\mathcal{F}_t,\ t \geq 0$, contains all P-null sets in $\mathcal{F}$. $W(t)$ represents the m-dimensional Brownian motion.

The SDDE (2.1) has a *drift coefficient* function $H : [0,T] \times C \to R^d$ and a *diffusion coefficient* function $G : [0,T] \times C \to R^{d\times m}$ satisfying the following

Hypothesis 2.1 *(i) H and G are Lipschitz on bounded sets of C uniformly with respect to the first variable, i.e. for each integer $n \geq 1$, there exists a constant $L_n > 0$ (independent of $t \in [0,T]$) such that*

$$|H(t,\eta_1) - H(t,\eta_2)| + |G(t,\eta_1) - G(t,\eta_2)| \leq L_n \|\eta_1 - \eta_2\|$$

for all $t \in [0,T]$ and $\eta_1, \eta_2 \in C$ with $\|\eta_1\| \leq n$, $\|\eta_2\| \leq n$.

(ii) There is a constant $K > 0$ such that

$$|H(t,\eta)| + |G(t,\eta)| \leq K(1 + \|\eta\|)$$

for all $t \in [0,T]$ and $\eta \in C$.

A *solution* of (2.1) is a measurable, sample-continuous process $x : [-\tau, T] \times \Omega \to R^d$ such that $x|_{[0,T]}$ is $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted and x satisfies (2.1) almost surely.

In [50] it was shown that if Hypothesis 2.1 holds then for each $\phi \in C$ the SDDE

(2.1) has a unique solution $x^\phi : [-\tau, \infty) \times \Omega \to R^d$ with $x^\phi|_{[-\tau,0]} = \phi \in C$ and $[0,T] \ni t \to x_t^\phi \in C$ being sample-continuous.

2.2 Model of (B, S)-securities market with delayed response

In our model, the *bond* (risk-less asset) is represented by the price function $B(t)$ given by

$$B(t) = B_0 e^{rt}, \ t \in [0,T], \tag{2.2}$$

where $r > 0$ is the risk-free rate of return, and the *stock* (risky asset) is the stochastic process $(S(t))_{t\in[-\tau,T]}$ which satisfies the following one-dimensional SDDE

$$dS(t) = \mu S(t)\ dt + \sigma(t, S_t)S(t)\ dW(t), \tag{2.3}$$

where $S_t(\theta) := S(t+\theta)$, $\theta \in [-\tau, 0]$, $\mu \in R$, $\sigma : [0,T] \times C \to R$ is a continuous mapping, C is the Banach space of continuous functions from $[-\tau, 0]$ into R, equipped with the supremum norm, and $W(t)$ is a standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$ for which the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and each $\mathcal{F}_t$ with $t \geq 0$ contains all $\mathcal{P}$-null sets in $\mathcal{F}$. To specify a solution, we need to give the initial data of S on $[-\tau, 0]$. In this thesis, we assume this initial data are deterministic, that is, the *initial data* for (2.3) are given by $S(\theta) = \varphi(\theta)$ with $\theta \in [-\tau, 0]$ for some $\varphi \in C$.

The existence and uniqueness of a solution of (2.3) are guaranteed if the coefficients in (2.3) satisfy the following local Lipschitz and growth conditions:

$$\begin{gathered}\forall n \geq 1 \quad \exists L_n > 0 \quad \forall t \in [0,T] \quad \forall \eta_1, \eta_2 \in C, \; \|\eta_1\| \leq n, \; \|\eta_2\| \leq n : \\ |\sigma(t,\eta_1)\,\eta_1(0) - \sigma(t,\eta_2)\,\eta_2(0)| \leq L_n \|\eta_1 - \eta_2\|\end{gathered} \tag{2.4}$$

and

$$\exists K > 0 \quad \forall t \in [0,T], \; \eta \in C : \quad |\sigma(t,\eta)\,\eta(0)| \leq K(1 + \|\eta\|). \tag{2.5}$$

The *discounted stock price* is defined by

$$Z(t) := \frac{S(t)}{B(t)}. \tag{2.6}$$

Using Girsanov's theorem (see [46]), we obtain the following result concerning the change of probability measure in the above market.

Lemma 2.1 *For a given process $(S(t))_{[-\tau,T]}$, under the assumptions*

$$\int_0^T \left(\frac{r-\mu}{\sigma(t,S_t)} \right)^2 dt < \infty, \quad a.s. \tag{2.7}$$

$$E \exp \left\{ \frac{1}{2} \int_0^T \left(\frac{r-\mu}{\sigma(t,S_t)} \right)^2 dt \right\} < \infty \tag{2.8}$$

the following holds:

1) There is a probability measure $\mathcal{P}^*$ equivalent to $\mathcal{P}$ such that

$$\frac{d\mathcal{P}^*}{d\mathcal{P}} := \exp \left\{ \int_0^T \frac{r-\mu}{\sigma(s,S_s)}\, dW(s) - \frac{1}{2} \int_0^T \left(\frac{r-\mu}{\sigma(s,S_s)} \right)^2 ds \right\} \tag{2.9}$$

is its Radon-Nikodym density;

2) The discounted stock price $Z(t)$ is a positive local martingale with respect to $\mathcal{P}^*$, *and is given by*

$$Z(t) = Z_0 \exp\left\{-\frac{1}{2}\int_0^t \sigma^2(s, S_s)\, ds + \int_0^t \sigma(s, S_s)\, dW^*(s)\right\}, \tag{2.10}$$

where $W^*(t) := \int_0^t \frac{\mu - r}{\sigma(s, S_s)}\, ds + W(t)$ *is a standard Wiener process with respect to* $\mathcal{P}^*$.

Remarks: 1. The process $Z(t)$ can also be written as

$$dZ(t) = Z(t)\sigma(t, S_t)\, dW^*(t),$$

and, in particular, we have

$$d\ln Z(t) = -\frac{1}{2}\sigma^2(t, S_t)\, dt + \sigma(t, S_t)\, dW^*(t).$$

2. (2.8) is a sufficient condition for the right-hand side of (2.9) to be martingale with t in place of T.

3. Conditions (2.7)-(2.8) in the lemma are satisfied if there exists $\delta > 0$ such that $\sigma(t, \phi) \geq \delta$ for all $t \in [0, T]$ and $\phi \in C$.

Accordingly, the only source of randomness in our model for the market consisting of the stock $S(t)$ and the bond $B(t)$ is a standard Wiener process $W(t)$, $t \in [0, T]$, with T denoting the terminal time. This Wiener process generates the filtration $\mathcal{F}_t := \sigma\{W(s) : 0 \leq s \leq t\}$. It can be shown that the $\mathcal{P}^*$-completed filtrations generated by either W, W^*, S or Z all coincide. This is useful since S is the observed process. See [31] and [39] for details.

A process $\pi = (\alpha_t, \beta_t)_{t\in[0,T]}$ is called a *trading strategy* in α_t shares of bond and β_t shares of stock if π is predictable and $\left(\int_0^t \beta_s^2\, d[Z, Z]_s\right)^{\frac{1}{2}}$, $t \in [0, T]$, is locally integrable under $\mathcal{P}^*$, where $[Z,Z]_t$ is the quadratic variation. The strategy π is *admissible* if it is *self-financing*, i.e. the *discounted value process* $X_t(\pi) := \alpha_t + \beta_t Z(t)$ solves

$$X_t(\pi) = X_0(\pi) + \int_0^t \beta \, dZ,$$

and if, in addition, $X_t(\pi)$ is a nonnegative martingale under $\mathcal{P}^*$. A *contingent claim* $\mathcal{C}$ is a positive $\mathcal{F}_T$-measurable random variable. We call a contingent claim *attainable* if there exists an admissible strategy π that *generates* $\mathcal{C}$, i.e. $X_T(\pi) = e^{-rT}\mathcal{C}$. For such a claim $\mathcal{C}$, $\phi_0 := X_0(\pi) = E_{\mathcal{P}^*}(e^{-rT}\mathcal{C})$ is called the *price associated with* $\mathcal{C}$ and this is the only reasonable price for $\mathcal{C}$ at time 0 if we assume the absence of arbitrage opportunities. For times t between 0 and T, the fair price of the claim is given by $\phi_t = e^{rt}X_t(\pi) = E_{\mathcal{P}^*}(e^{-r(T-t)}\mathcal{C} \mid \mathcal{F}_t)$. A market is said to be *complete* if every $\mathcal{P}^*$-integrable claim is attainable.

Theorem 2.1 *(Completeness)*
(i) If for any given $S(t)$ the discounted stock price process $Z(t)$ is a martingale under $\mathcal{P}^*$, then the model (2.3) *is complete;*
(ii) Under condition (2.7) *for every given $S(t)$, the model* (2.3) *is complete and the initial price of any integrable claim $\mathcal{C}$ is given by*

$$\phi_0 = E_{\mathcal{P}^*}(e^{-rT}\mathcal{C}), \tag{2.11}$$

and the price of the claim at any time $0 \le t \le T$ is given by $\phi_t = E_{\mathcal{P}^*}(e^{-r(T-t)}\mathcal{C} \mid \mathcal{F}_t)$.

The proof of this theorem is standard and is similar, for example, to the proof of corresponding theorems in [31] and [39].

Let the so-called *market price of risk* process be given by $\lambda(t) := \frac{\mu - r}{\sigma(t,S_t)}$ for $t \ge 0$.

Changing the probability measure in equation (2.3) for stock price and using Ito's lemma lead to

$$\ln S(t) = \ln S(0) + \int_0^t (r - \frac{1}{2}\sigma^2(u, S_u))\ du + \int_0^t \sigma(u, S_u)\ dW^*(u),$$

or, equivalently,

$$\ln \frac{S(t)}{S(t-\tau)} = r\tau - \frac{1}{2}\int_{t-\tau}^t \sigma^2(u, S_u)\ du + \int_{t-\tau}^t \sigma(u, S_u)\ dW^*(u), \tag{2.12}$$

where $W^*(t) = \int_0^t \lambda(s)\ ds + W(t)$. The expression (2.12), as well as the following expressions in terms of the physical measure $\mathcal{P}$, will be needed later for deriving a continuous-time analogue of GARCH(1,1)-model for stochastic volatility:

$$\ln \frac{S(t)}{S(t-\tau)} = r\tau + \int_{t-\tau}^t \left[\lambda(u)\sigma(u, S_u) - \frac{1}{2}\sigma^2(u, S_u)\right]\ du + \int_{t-\tau}^t \sigma(u, S_u)\ dW(u). \tag{2.13}$$

We conclude this subsection by showing that $S(t) > 0$ a.s. for all $t \in [0, T]$, when $\varphi(0) > 0$. Define the following process:

$$N(t) := \mu t + \int_0^t \sigma(s, S_s)\ dW(s), \quad t \in [0, T].$$

This is a semimartingale with the quadratic variation $\langle N \rangle(t) = \int_0^t \sigma^2(s, S_s)\ ds$. Then, from equation (2.3) we get

$$dS(t) = S(t)\ dN(t), \quad S(0) = \varphi(0).$$

This equation has a solution:

$$\begin{aligned} S(t) &= \varphi(0)\exp\left\{N(t) - \frac{1}{2}\langle N\rangle(t)\right\} \\ &= \varphi(0)\exp\left\{\mu t + \int_0^t \sigma(u, S_u)\, dW(u) - \frac{1}{2}\int_0^t \sigma^2(u, S_u)\, du\right\}. \end{aligned}$$

From this we see that if $\varphi(0) > 0$, then $S(t) > 0$ a.s. for all $t \in [0, T]$.

2.3 A continuous-time GARCH model

In this section, we show that a model of (B, S)-securities market with delayed response arises as a continuous-time equivalent of GARCH(1,1)-model. The GARCH models are proved consistent with the stock market data and are widely used in equity modeling (see [10]).

We continue to consider the risk-neutral world where the stock price $S(t)$ has the dynamics given by

$$dS(t) = rS(t)\, dt + \sigma(t, S_t)S(t)\, dW^*(t),$$

where $W^*(t)$ is defined in Lemma 2.1 and $S_t(\theta) = S(t+\theta), \theta \leq 0$. We consider the following equation for the variance $\sigma^2(t, S_t)$:

$$\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^t \sigma(s, S_s)\, dW(s)\right]^2 - (\alpha+\gamma)\sigma^2(t, S_t). \tag{2.14}$$

Here, all the parameters α, γ, τ, V are positive constants. The Wiener process $W(t)$ is the same as in (2.3). Note that the solution of (2.14) depends on the history of $S(t)$ from time 0, which does not fit the framework of (2.3). Considering σ^2 a variable makes equations (2.3),(2.14) form a 2-dimensional system of SDDEs and the Hypothesis 2.1 applies. Later, in Section 3.2, we consider a simplified version (3.12) of (2.14) that fits into framework (2.3). This framework is particularly convenient when deriving an option pricing PDE (see Section 3.1).

Note that our model is different from the continuous-time analogue of GARCH model given in [52]. The latter one is sometimes called GARCH diffusion, mainly because of another Wiener process appearing in the equation for volatility. However, ours is more in line with the original spirit of GARCH, since it has a longer "memory" in the volatility term. And most importantly, our model contains only one source of randomness, i.e. the Wiener process in the equation for stock price (for derivation see Section 6.1 of the Appendices).

Taking into account (2.13), we note that equation (2.14) is equivalent to

$$\begin{aligned}\frac{d\sigma^2(t,S_t)}{dt} =&\gamma V + \frac{\alpha}{\tau}\left[\ln\frac{S(t)}{S(t-\tau)} - r\tau - \int_{t-\tau}^{t}(\lambda(u)\sigma(u,S_u) - \frac{1}{2}\sigma^2(u,S_u))\,du\right]^2 \\ &- (\alpha+\gamma)\sigma^2(t,S_t).\end{aligned} \tag{2.15}$$

Using risk-neutral measure argument, we obtain from (2.14) that

$$\begin{aligned}&\frac{d\sigma^2(t,S_t)}{dt} = \\ &= \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t}\sigma(s,S_s)\,dW^*(s) - \int_{t-\tau}^{t}\lambda(u)\sigma(u,S_u)\,du\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t) \\ &= \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t}\sigma(s,S_s)\,dW^*(s) - (\mu-r)\tau\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t).\end{aligned}$$

Taking the expectations under risk-neutral measure $\mathcal{P}^*$ on both sides of the equation above, and denoting $v(t) := E_{\mathcal{P}^*}[\sigma^2(t, S_t)]$, we obtain the following deterministic delay differential equation

$$\frac{dv(t)}{dt} = \gamma V + \alpha\tau(\mu - r)^2 + \frac{\alpha}{\tau}\int_{t-\tau}^{t} v(s)\, ds - (\alpha+\gamma)v(t). \tag{2.16}$$

Both the stochastic process $\sigma^2(t, S_t)$ and the deterministic process $v(t)$ have the same initial data $\sigma_0^2(t)$ on the interval $[-\tau, 0]$:

$$\sigma^2(t) = v(t) = \sigma_0^2(t), \quad t \in [-\tau, 0].$$

Note that (2.16) has a stationary solution $v(t) \equiv X = V + \alpha\tau(\mu - r)^2/\gamma$.

An unusual result is that equation (2.16) for the expectation of the squared volatility under the risk-neutral measure $\mathcal{P}^*$ contains the drift parameter μ. In standard equity option pricing problems the drift parameter plays no role, disappeared through the Girsanov transformation (see Lemma 2.1). However, our model inherited this property from the discrete-time GARCH(1,1) model where the drift parameter enters the equation for volatility:

$$\begin{aligned} \ln(S_n/S_{n-1}) &= m + \sigma_n\xi_n, \quad \{\xi_n\} \sim \text{i.i.d. } N(0,1), \\ \sigma_n^2 &= \gamma V + \alpha\left(\sigma_{n-1}\xi_{n-1}\right)^2 + (1-\alpha-\gamma)\sigma_{n-1}^2 \\ &= \gamma V + \alpha\left(\ln(S_{n-1}/S_{n-2}) - m\right)^2 + (1-\alpha-\gamma)\sigma_{n-1}^2. \end{aligned}$$

It seems nontrivial to have an explicit formula for a solution of (2.16) with arbitrarily given initial data. However we can describe asymptotic behaviors of solutions of (2.16) by substituting $v(t) = X + Ce^{\rho t}$ into (2.16) to obtain the so-called characteristic equation for ρ (see [30])

$$\rho^2 = \frac{\alpha}{\tau} - \frac{\alpha}{\tau}e^{-\rho\tau} - (\alpha+\gamma)\rho.$$

The only non-zero solution to this equation is $\rho \approx -\gamma$. Then, we have $v(t) \approx X + Ce^{-\gamma t}$ for large values of t, and X is asymptotically stable.

These observations can be directly checked using numerical simulations for equation (2.16). The numerical scheme is defined as follows.

$$v_i = \gamma X \Delta t + (1 + \frac{\alpha(\Delta t)^2}{\tau} - (\alpha+\gamma)\Delta t)v_{i-1} + \frac{\alpha(\Delta t)^2}{\tau}(v_{i-2} + ... + v_{i-l}),$$

where $v_i = v(t_i)$ and $\{t_i\}$ is a time grid with a mesh of constant size Δt. A typical solution is shown in Figure 1. Figure 2 shows the dependence of the terminal expected variance $v(T)$ on delay τ for a typical constant initial value.

Also see Table 1 for implied volatility structure in the market with volatility described by (2.14). If the data in the table were plotted against the strike price we could see the U-shaped plot. Note however that the market option prices are observed to have a similar pattern. This is an important observation in support of our continuous-time GARCH model.

3 Option pricing and numerical simulations

In this Chapter, we provide a general framework for pricing European call options written on a stock whose volatility is a continuous function of time and the path S_t of the stock price process. By deriving an analogue of Ito's lemma, we obtain an integro-differential equation for a function of the option price with boundary conditions specified according to the type of option to be priced. We solve this equation using a numerical scheme obtained through a finite-difference approximation of derivatives. We also provide an alternative way to price options through accelerated Monte Carlo simulations.

3.1 General option pricing framework

The stock price value is assumed to satisfy the following SDDE:

$$dS(t) = rS(t)dt + \sigma(t, S_t)S(t)dW(t) \tag{3.1}$$

with continuous deterministic initial data $S_0 = \varphi \in C := C([-\tau, 0], R)$, where σ represents a *volatility*, which is a continuous function of time and the elements of C. Note that the Wiener process W corresponds to W^* from the previous Chapter.

As it was mentioned in the Section 2.1, the existence and uniqueness of a solution of (3.1) are guaranteed if the coefficients in (3.1) satisfy the following local Lipschitz and growth conditions:

$$\begin{gathered} \forall n \geq 1 \quad \exists L_n > 0 \quad \forall t \in [0, T] \quad \forall \eta_1, \eta_2 \in C, \ \|\eta_1\| \leq n, \ \|\eta_2\| \leq n : \\ |\sigma(t, \eta_1)\ \eta_1(t) - \sigma(t, \eta_2)\ \eta_2(t)| \leq L_n \|\eta_1 - \eta_2\| \end{gathered} \tag{3.2}$$

and

$$\exists K > 0 \quad \forall t \in [0,T],\ \eta \in C: \quad |\sigma(t,\eta)\ \eta(t)| \leq K(1 + \|\eta\|). \tag{3.3}$$

Note that the stock price values are positive with probability 1 if the initial data are positive, that is, $\varphi(\theta) > 0$ for all $\theta \in [-\tau, 0]$.

As shown in Section 2.2, the fair price of a European call option with maturity T and strike price K is given by the conditional expectation of the discounted final payoff

$$F(t) = E\left[e^{-r(T-t)} \max(S(T) - K, 0) | \mathcal{F}_t\right].$$

More generally, if $\mathcal{C}$ is any contingent claim, then the associated option price is $F(t) = E[e^{-r(T-t)}\mathcal{C} \mid \mathcal{F}_t]$. Now using the Markov property of solutions of SDDEs (see [50]), we deduce that this expectation is a functional of S_t (rather than a function of $S(t)$, which is true for stochastic ODEs). Therefore, the option price is given by a functional $F(t, S_t)$.

We are primarily interested in an option price value, which is assumed to depend on the current and the previous stock price values in the following way:

$$F(t, S_t) = \int_{-\tau}^{0} e^{-r\theta} H(S(t+\theta), S(t), t) d\theta, \tag{3.4}$$

where $H \in C^{0,2,1}(R \times R \times R_+)$. Such a representation is chosen since it includes sufficiently general functionals for which an analogue of Ito's lemma can be derived. If H is chosen appropriately, these correspond to option prices for the contingent claims $\mathcal{C} = F(T, S_T)$. We will need to derive conditions on H for F to be such an option price. Note that even if a European call option does not satisfy condition, we may still be able to precisely price other options that approximate this call.

Lemma 3.1 *Suppose a functional $F : R_+ \times C \to R$ has the following form*

$$F(t, S_t) = \int_{-\tau}^{0} h(\theta) H(S_t(\theta), S_t(0), t) d\theta,$$

for $H \in C^{0,2,1}(R \times R \times R_+)$ and $h \in C^1([-\tau, 0], R)$. Then

$$F(t, S_t) = F(0, \varphi) + \int_0^t \mathcal{A}F(s, S_s) ds + \int_0^t \sigma(s, S_s) S(s) \mathcal{B}F(s, S_s) dW(s), \qquad (3.5)$$

where for $(t, x) \in R_+ \times C$,

$$\begin{aligned} \mathcal{A}F(t, x) =& h(0) H(x(0), x(0), t) - h(-\tau) H(x(-\tau), x(0), t) - \\ & - \int_{-\tau}^{0} h'(\theta) H(x(\theta), x(0), t) d\theta + \int_{-\tau}^{0} h(\theta) L H(x(\theta), x(0), t) d\theta, \\ \mathcal{B}F(t, x) =& \int_{-\tau}^{0} h(\theta) H_2'(x(\theta), x(0), t) d\theta, \end{aligned}$$

and

$$\begin{aligned} LH(x(\theta), x(0), t) =& r x(0) H_2'(x(\theta), x(0), t) + \frac{\sigma^2(t, x) x^2(0)}{2} H_{22}''(x(\theta), x(0), t) + \\ & + H_3'(x(\theta), x(0), t), \end{aligned}$$

where $H_i', i = 1, 2, 3$, represents the derivative of $H(x(\theta), x(0), t)$ with respect to i-th argument.

Proof: We defer the proof to the Section 6.2 of the Appendices.

In what follows, we assume that a risk-less portfolio consisting of a position in the option and a position in the underlying stock is set up. In the absence of arbitrage opportunities, the return from the portfolio must be risk-free with the spot rate

r. The portfolio Π has to be risk-less during the time interval $[t, t+dt]$ and must instantaneously earn the same rate of return as other short-term risk-free securities. It follows that $d\Pi(t) = r\Pi(t)dt$ and this will be used in the proof of the following theorem.

Theorem 3.1 *Suppose the functional F given by* (3.4) *with $S(t)$ satisfying* (3.1) *is an option price and $H \in C^{0,2,1}(R \times R \times R_+)$. Then, $H(S(t+\theta), S(t), t)$ satisfies the following equation*

$$0 = H|_{\theta=0} - e^{-r\theta} H|_{\theta=-\tau} + \int_{-\tau}^{0} e^{-r\theta} \left(H_3' + rS(t)H_2' + \frac{1}{2}\sigma^2(t, S_t)S^2(t)H_{22}'' \right) d\theta \quad (3.6)$$

for all $t \in [0, T]$.

Proof: To construct an equation for F, we need to consider a portfolio which consists of -1 derivative and $\mathcal{B}F(t, S_t)$ shares. Then, the portfolio value $\Pi(t)$ is equal to

$$\Pi(t) = -F(t, S_t) + \mathcal{B}F(t, S_t)\; S(t),$$

and the associated infinitesimal change in the time interval $[t, t+dt]$ is

$$d\Pi = -dF + \mathcal{B}F\; dS.$$

We should point out here that in the last statement we suppose that $(\mathcal{B}F)$ is held constant during the time-step dt, and hence term $d(\mathcal{B}F)$ is equal to zero. If this were not the case then $d\Pi$ would contain term $d(\mathcal{B}F)$.

Using (3.5) and (3.1), we obtain

$$d\Pi = -\mathcal{A}F\,dt - \sigma S\mathcal{B}F\,dW + \mathcal{B}F(rS\,dt + \sigma S\,dW).$$

Consideration of risk-free during the time dt then leads to

$$d\Pi = r\Pi\,dt,$$

that is,

$$-\mathcal{A}F(t,S_t) + rS(t)\mathcal{B}F(t,S_t) = r(-F(t,S_t) + \mathcal{B}F(t,S_t)S(t)),$$

or

$$\mathcal{A}F(t,S_t) = rF(t,S_t).$$

Therefore, the equation for $H(S(t+\theta),S(t),t)$ becomes

$$0 = H|_{\theta=0} - e^{-r\theta}H|_{\theta=-\tau} + \int_{-\tau}^{0} e^{-r\theta}\left(H_3' + rS(t)H_2' + \frac{1}{2}\sigma^2(t,S_t)S^2(t)H_{22}''\right)d\theta.$$

This completes the proof.

Remark: Solving equation (3.6), we can determine the trading strategy that replicates the option price. Therefore, our assumption of the option price in the form (3.4) holds whenever there is a solution to the equation. Later in Section 3.3, we construct the solution when the volatility is given by $\sigma(t,\varphi) \equiv \sigma(t,\varphi(0),\varphi(-\tau))$ for $\varphi \in C$. Note however that there is no solution for arbitrary choices of $\sigma(t,\varphi)$. This becomes evident when we pick some H and solve equation (3.6) for σ^2. Therefore, the class of models for which the price can be found is quite restricted but more general than $\sigma(t, \varphi(0), \varphi(-\tau))$. It does not contain, however, $\sigma(t, \varphi(0), \varphi(-\tau_1), \varphi(-\tau))$ for $\tau_1 \neq \tau$.

Consider a *European call option* with the final payoff $\max(S - K, 0)$ at the maturity time T (see [35]). Then problem (3.6) has the boundary condition at the time T

$$F(T, S_T) = \max(S(T) - K, 0), \tag{3.7}$$

which can be approximated by its functional analogue

$$F(T, S_T) = \frac{1}{\tau} \int_{-\tau}^{0} \max(e^{-r\theta} S_T(\theta) - K, 0) d\theta, \tag{3.8}$$

where $1/\tau$ is a normalizing factor such that $F(T, S_T) \to \max(S(T) - K, 0)$ as $\tau \to 0$.

3.2 A continuous-time GARCH model for volatility with bounded memory

In this section, we introduce another continuous-time version of GARCH(1,1) model. It has a connection with the model derived in Section 2. However, as we shall see later, the new one is more suitable for the option pricing framework introduced in the previous section.

Similarly to (2.14), assume $\sigma^2(t)$ satisfies the following equation

$$\frac{d\sigma^2(t)}{dt} = \gamma V + \frac{\alpha}{\tau} \ln^2 \left(\frac{S(t)}{S(t-\tau)} \right) - (\alpha + \gamma)\sigma^2(t), \tag{3.9}$$

where V is a long-run average variance rate, α and γ are positive constants. Here,

$S(t)$ is a solution of the SDDE (3.1) with positive initial data $\varphi \in C$.

Consider a grid $-\tau = t_{-l} < t_{-l+1} < \cdots < t_0 = 0 < t_1 < \cdots < t_N = T$ with the time step size Δ_l of the form

$$\Delta_l = \frac{\tau}{l},$$

where $l \geq 2$. Then a discrete time analogue of (3.9) is

$$\sigma_n^2 = \gamma V + \frac{\alpha}{l} \ln^2(S_{n-1}/S_{n-1-l}) + (1 - \alpha - \gamma)\sigma_{n-1}^2,$$

where $\sigma_n^2 = \sigma^2(t_n)$ and $S_n = S(t_n)$. Note that the process described by this difference equation is a generalization of the GARCH(1,1) (with returns assumed to have conditional mean zero),

$$\sigma_n^2 = \gamma V + \alpha \ln^2(S_{n-1}/S_{n-2}) + (1 - \alpha - \gamma)\sigma_{n-1}^2. \tag{3.10}$$

Now, using a variation of constants formula for (3.9) we obtain

$$\sigma^2(t) = \frac{\gamma V}{\alpha + \gamma} + \left(\sigma^2(t_0) - \frac{\gamma V}{\alpha + \gamma}\right) e^{-(\alpha+\gamma)(t-t_0)} + \frac{\alpha}{\tau} \int_{t_0}^{t} e^{(\alpha+\gamma)(\xi - t)} \ln^2\left(\frac{S(\xi)}{S(\xi - \tau)}\right) d\xi \tag{3.11}$$

for $t_0 \geq 0$. It is then natural that we consider the following expression for variance:

$$\bar{\sigma}^2(t) = \sigma^2(t_0)e^{-(\alpha+\gamma)(t-t_0)} + \left[\gamma V + \frac{\alpha}{\tau} \ln^2\left(\frac{S(t)}{S(t-\tau)}\right)\right] \frac{1 - e^{-(\alpha+\gamma)(t-t_0)}}{\alpha + \gamma}, \tag{3.12}$$

since functions $\bar{\sigma}^2$ and σ^2 are close to each other in the following sense:

$$\sigma^2(t) = \bar{\sigma}^2(t) + o(|t - t_0|) \quad \text{as } t \to t_0.$$

We shall use the model (3.12) as a model for volatility in option pricing that fits the framework (3.1).

3.3 Finite-difference method for the general equation

In this section, we show how to solve the general equation (3.6) for evaluation function H, which will allow us to find the value of European call option price in a (B, S)-market with delayed response (2.2)-(2.3). We only consider the case $\sigma(t, \varphi) = \sigma(t, \varphi(0), \varphi(-\tau))$. In all other cases, the technique here cannot be applied.

Let's consider a continuous function $\varphi \in C([-\tau, 0], R)$ with $\varphi(0) = x, \varphi(-\tau) = y$, $x, y \in R$. Then, the system (3.6) in terms of continuous function $S_t = \varphi$ will have the following form

$$\begin{aligned} &0 = H(x,x,t) - e^{r\tau} H(y,x,t) + \int_{-\tau}^{0} e^{-r\theta} \left(H'_t + rxH'_x + \frac{1}{2}\sigma^2(t,\varphi)x^2 H''_{xx} \right) d\theta, \\ &\int_{-\tau}^{0} e^{-r\theta} H(\varphi(\theta), x, T)\, d\theta = \max(x-K, 0). \end{aligned} \tag{3.13}$$

Our main objective now is to solve system (3.13) for the function $H(y, x, t)$. Let us consider the function $\varphi_{xy}(\theta) = x + (y - x)(e^{-r\theta} - 1)/(e^{r\tau} - 1)$, $\theta \in [-\tau, 0]$, which connects y and x. After substituting φ_{xy} into (3.13) and changing the integration over the variable θ to the variable $s = \varphi_{xy}(\theta)$, we obtain the following:

$$\begin{aligned} 0 = &\frac{x-y}{\hat{\tau}} \left(H(x,x,t) - e^{r\tau} H(y,x,t) \right) + \\ &\quad + \int_{y}^{x} \left(H'_t + rxH'_x + \frac{1}{2}\sigma^2(t,\varphi_{xy})x^2 H''_{xx} \right)(s,x,t)\, ds, \\ &\int_{y}^{x} H(s,x,T) ds = \frac{1}{\hat{\tau}}(x-y)\max(x-K,0), \end{aligned} \tag{3.14}$$

where $\hat{\tau} = (e^{r\tau} - 1)/r$. Note that equation (3.14) is equivalent to equation (3.13) if functional the $\sigma(t, \varphi)$ is a function of t, $\varphi(0)$ and $\varphi(-\tau)$, that is when

$$\sigma(t, \varphi) \equiv \sigma(t, \varphi(0), \varphi(-\tau)).$$

Note, however, that the volatility given by (3.12) satisfies this condition.

Equation (3.14) is an integro-differential equation and it can be reduced to a PDE using the following substitution:

$$f(x, y, t) = \int_y^x H(s, x, t)\, ds,$$

and the PDE has the following form:

$$\begin{aligned} 0 = \frac{x-y}{\hat{\tau}} \left(-\frac{\partial f}{\partial y}|_{y=x} + e^{r\tau} \frac{\partial f}{\partial y} \right) + \frac{\partial f}{\partial t} + rx \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}|_{y=x} \right) + \\ + \frac{1}{2} \sigma^2(t, x, y) x^2 \left(\frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial^2 f}{\partial x \partial y}|_{y=x} + \frac{\partial^2 f}{\partial y^2}|_{y=x} \right) \end{aligned} \tag{3.15}$$

subject to boundary conditions

$$\begin{aligned} f|_{t=T} &= \frac{1}{\hat{\tau}}(x - y) \max(x - K, 0), \\ f|_{y=x} &= 0. \end{aligned}$$

An analytic solution to equation (3.15) seems difficult to find. One way to solve it is to consider the finite-difference numerical approximation scheme for derivatives in (3.15). We obtain the following iterative updating scheme as we move back in time from T:

$$\begin{aligned} f_{i,j}^{\text{new}} &= (1-2a_{i,j})f_{i,j} + c_{i,j}[f_{i,j+1} - f_{i,j-1}] + (b_i + a_{i,j})f_{i+1,j} \\ &\quad + (-b_i + a_{i,j})f_{i-1,j} + (-d_{i,j} + b_i + a_{i,j})f_{i,i+1} \\ &\quad + (d_{i,j} - b_i + a_{i,j})f_{i,i-1} + (-a_{i,j}/2)[f_{i+1,i-1} + f_{i-1,i+1}], \end{aligned} \tag{3.16}$$

where the coefficients are defined by

$$\begin{aligned} a_{i,j} &= \sigma_{i,j}^2 x_i^2 \frac{\Delta t}{2(\Delta x)^2}, \qquad & b_i &= r x_i \frac{\Delta t}{2\Delta x}, \\ c_{i,j} &= e^{r\tau} \frac{x_i - x_j}{\hat{\tau}} \frac{\Delta t}{2\Delta x}, & d_{i,j} &= \frac{x_i - x_j}{\hat{\tau}} \frac{\Delta t}{2\Delta x}, \end{aligned}$$

$f_{i,j} = f(x_i, x_j, t)$ and $f_{i,j}^{\text{new}} = f(x_i, x_j, t - \Delta t)$. The scheme (3.16) seems stable as $(\Delta t, \Delta x) \to 0$ if the following condition holds:

$$\sigma_{i,j}^2 x_i^2 \frac{\Delta t}{(\Delta x)^2} < 1.$$

See Table 2 and 3 for numerical results on the finite-difference method (3.16) applied to continuous-time GARCH model (3.12). The algorithm is given at the end of this thesis as Program 1.

3.4 Option pricing via accelerated Monte Carlo

Here we review some facts from Section 3.1 on option pricing in the market where the stock price follows (3.1) with the volatility given by (3.12). The fair price of the European option with terminal payoff $g(S(T))$ is given by the following conditional expectation

$$E\left[e^{-r(T-t)} g(S(T)) \mid \mathcal{F}_t\right]. \tag{3.17}$$

Using the Markov property for (3.1), the expectation is a functional of $S_t := \{S(u) : t-\tau \le u \le t\}$, which we denote $F(t, S_t)$.

We seek $F(t, S_t)$ in the following form

$$F(t, S_t) = \int_{-\tau}^{0} e^{-r\theta} H(S(t+\theta), S(t), t) d\theta, \tag{3.18}$$

where $H \in C^{0,2,1}(R \times R \times R_+)$. Then, $H(S(t+\theta), S(t), t)$ satisfies equation (3.6):

$$0 = L(t, S_t) \equiv$$
$$H|_{\theta=0} - e^{-r\theta} H|_{\theta=-\tau} + \int_{-\tau}^{0} e^{-r\theta} \left(H_3' + rS(t)H_2' + \frac{1}{2}\sigma^2(t, S_t)S^2(t)H_{22}'' \right) d\theta.$$

To find the option price $F(t, S_t)$, we can solve the equation subject to boundary condition

$$F(T, S_T) = g(S(T)).$$

A solution to equation (3.6) seems hard to find in a closed form. However, we can employ a finite difference scheme to solve the equation numerically (see Section 3.3).

On the other hand, we can use Monte Carlo simulation of independent realizations $S^{(n)}(t)$ of the process $S(t)$ (discretized using the Euler scheme, see [45]) and approximate the expectation (3.17) with

$$F(t, S_t) \approx \frac{1}{N} \sum_{n=1}^{N} e^{-r(T-t)} g(S^{(n)}(T)). \tag{3.19}$$

Then, by the central limit theorem, the option price $F(t, S_t)$ belongs to a confidence interval whose radius is proportional to the square root variance of the estimator. Normally, the variance is large and a substantial number of simulations are required to obtain a desired precision.

Both methods of finding the option price are more or less equivalent in terms of the computational efficiency. However, we can increase the efficiency in the second approach by reducing the variance of the estimator. With reduced variance, a smaller simulation time would be needed to get the desired precision of the estimator (3.19).

There is an efficient variance reduction technique, the so-called importance sampling for diffusions (see [53]). Employing the importance sampling variance reduction technique requires some approximation of the function $F(t, S_t)$. Using equation (3.6), we can approximate the function and then use this approximation to derive a more efficient Monte Carlo estimator similar to (3.19).

3.4.1 Importance sampling for diffusions with delay

Here we adapt a general formulation of the importance sampling technique introduced in [53]. Given a scalar square integrable $\mathcal{F}_t$-adapted process of the form $h(t, S_t)$, we define the following process

$$Q(t) = \exp\left\{\int_0^t h(u, S_u)\, dW(u) + \frac{1}{2}\int_0^t h^2(u, S_u)\, du\right\}.$$

If $E\left[Q(t)^{-1}\right] = 1$, then $(Q(t))_{0\leq t\leq T}$ is a positive martingale and we can define an equivalent to $\mathcal{P}^*$ probability measure $\mathcal{Q}$ through Radon-Nikodym density

$$\frac{d\mathcal{Q}}{d\mathcal{P}^*} = Q(T)^{-1}.$$

By Girsanov's theorem, the process defined by

$$W^{\mathcal{Q}}(t) = W(t) + \int_0^t h(u, S_u)\, du$$

is a standard Wiener process under the measure $\mathcal{Q}$. With respect to this new measure, the option price defined by $F(t, S_t)$ can be written

$$F(t, S_t) = E^{\mathcal{Q}}\left[e^{-r(T-t)} g(S(T))Q(T) \mid \mathcal{F}_t\right].$$

We can estimate the expectation by

$$\frac{1}{N}\sum_{n=1}^{N} e^{-r(T-t)} g(S^{(n)}(T))Q^{(n)}(T), \tag{3.20}$$

whose variance may be smaller than the variance of the estimator (3.19). Determining the function $h(t, S_t)$ that makes the variance smaller (or the smallest possible) is the sole goal of the *importance sampling method.*

The stock price process S satisfies the following equation in terms of the new Wiener process $W^{\mathcal{Q}}$

$$dS(t) = [r - \sigma(t)h(t, S_t)]\, S(t)\; dt + \sigma(t)S(t)\; dW^{\mathcal{Q}}(t).$$

Using Lemma 3.1, we can write an equation for $F(t, S_t)$

$$dF(t, S_t) = [L(t, S_t) + rF(t, S_t)]\;\; dt + \sigma(t)S(t) \int_{-\tau}^{0} e^{-r\theta} H_2'(S(t+\theta), S(t), t)d\theta\; dW(t),$$

where operator $L(t, S_t)$ is defined by (3.6). Since $L(t, S_t) = 0$, we have

$$d(F(t,S_t)Q(t)) = rF(t,S_t)Q(t)\,dt+$$
$$+\left(F(t,S_t)h(t,S_t) + \sigma(t)S(t)\int_{-\tau}^{0} e^{-r\theta}H_2'(S(t+\theta),S(t),t)d\theta\right)Q(t)\,dW^Q(t),$$

which leads to the following expression for variance

$$\mathrm{Var}^Q(g(S(T))Q(T)) = E^Q\int_0^T e^{2r(T-t)}Q^2(t)\times$$
$$\times\left[F(t,S_t)h(t,S_t) + \sigma(t)S(t)\int_{-\tau}^{0} e^{-r\theta}H_2'(S(t+\theta),S(t),t)d\theta\right]^2 dt,$$

as opposed to

$$\mathrm{Var}^{\mathcal{P}^*}(g(S(T))) = E^{\mathcal{P}}\int_0^T e^{2r(T-t)}\left[\sigma(t)S(t)\int_{-\tau}^{0} e^{-r\theta}H_2'(S(t+\theta),S(t),t)d\theta\right]^2 dt.$$

Therefore, the function $h(t, S_t)$ that makes the variance of the estimator (3.20) zero is

$$h(t,S_t) = -\frac{\sigma(t)S(t)}{F(t,S_t)}\int_{-\tau}^{0} e^{-r\theta}H_2'(S(t+\theta),S(t),t)d\theta. \tag{3.21}$$

Note that we could make the estimator of $F(t, S_t)$ "perfect" only if we knew the exact expression for $F(t, S_t)$. Fortunately, it is still possible to get a "good" estimator by approximating the function $F(t, S_t)$ and thus reducing the variance of the original estimator.

3.4.2 Option price approximation and variance reduction

Consider a European call option with maturity T and payoff $g(S(T)) = \max(S(T) - K, 0)$. The fair price for this option at time t when the stock price process follows

GARCH model (3.12) is given by

$$F(t, S_t) = E\left[e^{-r(T-t)} \max(S(T) - K, 0) \mid \mathcal{F}_t\right].$$

This expectation can be estimated with (3.19), whose variance can be reduced using the importance sampling method. As mentioned in the previous section, we need to get an approximation to the option price in order to define a new measure that reduces the variance of estimator.

We approximate the option price $F(t, S_t)$ with the Black-Scholes call option price. Observe that the GARCH volatility (3.12) is mean reverting to the level $\sqrt{V}$. We can use this fact to approximate option price $F(t, S_t)$ with Black-Scholes price $F_{\text{BS}}(t, S(t))$, where constant volatility $\sqrt{V_{\text{BS}}}$ is used. Then similarly to (3.21), we choose function $h(t, S_t)$ as

$$h(t, S_t) = -\frac{\sigma(t)S(t)}{F_{\text{BS}}(t, S(t))} \frac{\partial F_{\text{BS}}(t, S(t))}{\partial S}.$$

Although, it seems that the derivation of h depended on the use of H, it is still useful in the general case of $F(t, S_t)$. See [22] and [23] for the applications of similar h to stochastic volatility models.

In Figure 3, we show how the radius of 95% confidence interval of the option price estimator (3.20) varies as different values of V_{BS} are chosen. Note that the minimum radius 0.0195 is reached at V_{BS} close to the long-run variance rate $V = 0.0141$. The corresponding radius for the regular estimator (3.19) is 0.1328, and thus we reduced the radius of the confidence interval 6.8 times. This is equivalent to 46.4 times reduction of a number of realizations required to get a desired precision of the estimator.

The efficiency of the variance reduction can be judged from the following empirical viewpoint. If we were to simulate a Monte Carlo estimator of the call option price when the stock price volatility is constant, we could choose the estimator with theoretically zero variance since the option price in this case is known and it is given by the Black-Scholes formula. However, there is a discretization error in approximating the stock price process, and therefore the variance of the estimator is not zero anymore but close to it. We can use this lowest possible variance for a given time discretization step as a benchmark for the variance reduction in our model (3.1), (3.12).

The radius of the confidence interval corresponding to the lowest possible variance in the constant volatility case is 0.0135. Since for our model it is 0.0195, the efficiency of the variance reduction seems to be very good. Moreover, this shows that using a relatively low-order approximation of the option price, we can get a significant variance reduction. This fact was observed in [23] and it is confirmed by our numerical results.

4 Parameter estimation

Here we develop an estimation technique for some parameters involved in the analogue of GARCH(1,1) model introduced in Section 2.3. These are drift coefficient μ, time delay τ, and weights α, β and γ. The technique involves Maximum Likelihood (ML) method in combination with Akaike information criterion (AICC) applied to the equity price data. This criterion is widely used in statistical inference for model selection. However, there is a parameter that cannot be estimated from the equity price data: the spot risk-free rate r that arises only in a risk-neutral evaluation. We estimate this parameter (or the yield curve) from market prices of options with different maturities. Our results show that the yield curve is not flat but can be fit with the Cox-Ingersoll-Ross (CIR) model. We estimate the parameters in CIR model using the least-squares method.

4.1 Drift estimation

Parameter μ is unobservable, but it can be estimated from observations of $S(u)$, $u \in [0, t]$. The maximum likelihood estimator of μ is given by (see [1] and [2])

$$\tilde{\mu}(t) = \frac{1}{t}\int_0^t S^{-1}(u)dS(u).$$

Or, in terms of discrete-time observations over an increasing time-grid:

$$\frac{1}{t}\int_0^t S^{-1}(u)dS(u) = \lim_{n\to+\infty}\sum_{j=1}^{2^n} S^{-1}((j-1)t2^{-n})\left[S(jt2^{-n}) - S((j-1)t2^{-n})\right].$$

The statistical properties of $\tilde{\mu}(t)$ can be easily derived. Namely, since

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma(t, S_t)dW(t),$$

we have

$$\tilde{\mu}(t) = \mu + \frac{1}{t}\int_0^t \sigma(s, S_s)dW(s),$$

and hence $\tilde{\mu}(t)$ is distributed as

$$N\left(\mu, \frac{1}{t^2}\int_0^t E_{\mathcal{P}}\sigma^2(u, S_u)du\right),$$

where the expectation $E_{\mathcal{P}}\sigma^2(u, S_u)$ can be found explicitly using (2.14). This means that $\tilde{\mu}(t)$ is unbiased and mean-square consistent at the sampling interval $[0, t]$ as $t \to +\infty$. We note that t plays the role of the "sample size" in its numerical meaning, while 2^n is the numerical "computational size".

For a sample set of parameters, we can compute the variance of the estimator of μ and therefore a confidence interval for the true parameter. In our calculations, it seems that the size of the confidence interval is very large (e.g. $[0.05, 0.10]$) even when the data span years of observations. This means that we have to either find a better estimator with smaller variance or show that the parameter μ does not affect the final outcome of our analysis (e.g. option price). The latter can be shown quite easily. In Table 6, we present numerically computed European call option prices for different values of parameters r and μ. Observe that the option prices are practically not affected by the parameter μ. Therefore, for the purposes of option pricing, we keep parameter μ fixed at some reasonable value, e.g. 0.05.

4.2 Time delay and other parameters estimation

In this section, we show that the maximum likelihood (ML) method can be used to estimate parameters α, β, γ and V. Parameter l takes discrete values and has to be treated differently. We show that Akaike information criterion (AICC) can be used to select l.

4.2.1 Consistency and asymptotic normality of the ML estimators

For the simplicity of presentation, in this subsection we use notation x_t for $x(t)$. Suppose that we observe sequence $\{y_t\}$ with

$$\begin{aligned} y_t &= \mu_0 + \varepsilon_{0t}, \quad \varepsilon_{0t} = \xi_t h_{0t}^{1/2}, \\ h_{0t} &= \omega_0 + \frac{\alpha_0}{l}\mathcal{E}_{0t-1}^2 + \beta_0 h_{0t-1}, \end{aligned} \tag{4.1}$$

where $\mathcal{E}_{0t-1} = \sum_{k=1}^{l} \varepsilon_{0t-k}$, $l \geq 1$ is a fixed integer, $\{\xi_t\}_{t\in Z}$ is i.i.d. N(0,1). Let $\mathcal{F}_t$ be the σ-algebra generated by $\{y_t, y_{t-1}, \dots\}$. We define the compact parameter space

$$\Theta \equiv \left\{\theta = (\mu, \omega, \alpha, \beta) \in [-m, m] \times [w^{-1}, w] \times [a, 1-a] \times [b, 1-b] : \ \alpha + \beta \leq 1\right\}$$

for some positive constants m, w, a and b. We assume that the true parameter $\theta_0 = (\mu_0, \omega_0, \alpha_0, \beta_0)$ is in the interior of Θ. For any parameter, $\theta \in \Theta$ we define

$$\begin{aligned} y_t &= \mu + \varepsilon_t, \\ \hat{h}_t &= \omega + \frac{\alpha}{l}\mathcal{E}_{t-1}^2 + \beta \hat{h}_{t-1}, \quad \mathcal{E}_{t-1} = \sum_{k=1}^{l} \varepsilon_{t-k}, \end{aligned} \tag{4.2}$$

with the initial data given by $\hat{h}_0 = \omega/(1-\beta)$ and $\{\hat{h}_t\}_{-l+1 \le t \le -1}$, chosen arbitrarily. This gives the convenient expression for the variance process

$$\hat{h}_t = \frac{\omega}{1-\beta} + \frac{\alpha}{l} \sum_{i=0}^{t-1} \beta^i \mathcal{E}_{t-i-1}^2.$$

Since the conditional distribution of $\{\xi_t\}$ is the standard normal, the log-likelihood function takes the form (ignoring constants)

$$\hat{L}_T(\theta) = \frac{1}{2T} \sum_{t=1}^{T} \hat{l}_t(\theta), \quad \text{where} \quad \hat{l}_t(\theta) \equiv -\left(\ln \hat{h}_t(\theta) + \frac{\varepsilon_t^2}{\hat{h}_t(\theta)} \right).$$

It will be convenient to work with the unobserved variance process

$$h_t = \frac{\omega}{1-\beta} + \frac{\alpha}{l} \sum_{i=0}^{\infty} \beta^i \mathcal{E}_{t-i-1}^2$$

and the unobserved log-likelihood function

$$L_T(\theta) = \frac{1}{2T} \sum_{t=1}^{T} l_t(\theta), \quad \text{where} \quad l_t(\theta) \equiv -\left(\ln h_t(\theta) + \frac{\varepsilon_t^2}{h_t(\theta)} \right).$$

The process $h_t(\theta)$ is the model of the conditional variance when the infinite past history of the data is observed.

Lemma 4.1 *For all $\theta \in \Theta$, the expectation*

$$E\left[\frac{\partial h_t}{\partial \theta} \frac{\partial h_t}{\partial \theta'} h_t^{-2}\right]$$

exists and is a positive definite matrix.

We refer to [47] for the proof.

Theorem 4.1 *$E[L_T(\theta)]$ is uniquely maximized at θ_0.*

Proof: Consider

$$E[l_t(\theta)] - E[l_t(\theta_0)] = E\left[\ln\frac{h_{0t}}{h_t} - \frac{\varepsilon_t^2}{h_t} + \frac{\varepsilon_{0t}^2}{h_{0t}}\right].$$

Since $\varepsilon_t^2 = \varepsilon_{0t}^2 + 2(\mu_0 - \mu)\varepsilon_{0t} + (\mu_0 - \mu)^2$, using the law of iterated expectations we obtain

$$E[l_t(\theta)] - E[l_t(\theta_0)] = E\left[\ln\frac{h_{0t}}{h_t} - \frac{h_{0t}}{h_t} + 1 - \frac{(\mu_0 - \mu)^2}{h_t}\right] \leq 0,$$

where the equality takes place when $\ln(h_{0t}/h_t) = 0$ a.s. and $\mu = \mu_0$. The former expression is equivalent to

$$(\theta - \theta_0)'\left(\frac{\partial h_t}{\partial \theta} h_t^{-1}\right)_{\theta=\theta^*} = 0 \quad \text{a.s.}$$

for some $\theta^* \in \Theta$, which occurs if and only if $\theta = \theta_0$ by Lemma 4.1. Therefore, $E[L_T(\theta)]$ is uniquely maximized at θ_0.

We can then state two theorems, and we again refer to [47] for details.

Theorem 4.2 *Let θ_T be the solution to $\max_{\theta\in\Theta} L_T(\theta)$ and $\hat{\theta}_T$ the corresponding solution to $\max_{\theta\in\Theta} \hat{L}_T(\theta)$. Then $\theta_T \to \theta_0$ and $\hat{\theta}_T \to \theta_0$ in probability as $T \to \infty$.*

We define the following matrices:

$$A_0 = -E\left[\frac{\partial^2 l_t(\theta_0)}{\partial\theta\partial\theta'}\right],$$
$$\hat{A}_T = \frac{1}{2T}\sum_{t=1}^{T}\hat{h}_t^{-2}\frac{\partial\hat{h}_t}{\partial\theta}\frac{\partial\hat{h}_t}{\partial\theta'} + \frac{1}{T}\sum_{t=1}^{T}\hat{h}_t^{-1}\frac{\partial\varepsilon_t}{\partial\theta}\frac{\partial\varepsilon_t}{\partial\theta'},$$
$$A_T = \frac{1}{2T}\sum_{t=1}^{T}h_t^{-2}\frac{\partial h_t}{\partial\theta}\frac{\partial h_t}{\partial\theta'} + \frac{1}{T}\sum_{t=1}^{T}h_t^{-1}\frac{\partial\varepsilon_t}{\partial\theta}\frac{\partial\varepsilon_t}{\partial\theta'},$$
$$A = \frac{1}{2}E\left[h_t^{-2}\frac{\partial h_t}{\partial\theta}\frac{\partial h_t}{\partial\theta'}\right] + E\left[h_t^{-1}\frac{\partial\varepsilon_t}{\partial\theta}\frac{\partial\varepsilon_t}{\partial\theta'}\right].$$

Theorem 4.3 *The following statements hold:*

(a) $\sqrt{T}(\hat{\theta}_T - \theta_0) \sim N(0, A_0^{-1})$ *asymptotically as* $T \to \infty$*;*

(b) Consistent estimator of A_0 *is given by* $\hat{A}_T$ *evaluated at* $\hat{\theta}_T$*.*

4.2.2 Numerical results

We now show how to use the maximal likelihood method to estimate the time delay. In particular, we choose values for the parameters that maximize the chance (or likelihood) of the data occurring, and then use chosen values of parameters to test how our model (of market with delayed response) works.

Recall that the discrete-time model for volatility in the market with delayed response is

$$\begin{aligned}\varepsilon_n &= y_n - \mu \equiv \ln\frac{S_n}{S_{n-1}} - \mu,\\ \sigma_n^2 &= \omega + \frac{\alpha}{l}\left(\sum_{i=1}^{l}\varepsilon_{n-i}\right)^2 + \beta\sigma_{n-1}^2,\end{aligned} \tag{4.3}$$

where $\alpha + \beta + \gamma = 1$ and $\omega = \gamma V$. Parameter μ can be eliminated by assigning

$\mu = (\sum_{k=1}^{N} y_k)/N$. Parameter $l \geq 1$ represents the delay. For $l = 1$, we obtain GARCH(1,1) model. The correspondence between continuous-time parameter of delay τ and its discrete-time analogue l is given by $\tau = l\Delta$, where Δ is the size of a mesh of the discrete-time grid. The probability distribution of ε_n conditional on information up to time $n-1$ is assumed to be normal.

The likelihood function is given by

$$L(\alpha, \beta, \omega, l) = \prod_{n=1}^{N} \left[\frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(\frac{-\varepsilon_n^2}{2\sigma_n^2} \right) \right],$$

where σ_n is the function of α, β, ω and l (parameter γ can be eliminated due to equality above). Our task is to maximize the product subject to constraints:

$$\alpha \geq 0, \quad \beta \geq 0, \quad l \geq 1,$$
$$\alpha + \beta < 1.$$

Taking logarithms, we see that this is equivalent to maximizing (l is fixed for now)

$$f(\alpha, \beta, \omega, l) = \sum_{n=1}^{N} \left[-\ln(\sigma_n^2) - \frac{\varepsilon_n^2}{\sigma_n^2} \right]$$

with σ_n^2, $n \geq l+1$, explicitly given by

$$\sigma_n^2 = \omega A_n(\beta) + \alpha B_n(\beta) + R_n(\beta),$$

where

$$A_n(\beta) = 1 + \beta + \beta^2 + ... + \beta^{n-l-1},$$
$$B_n(\beta) = v_{n-1} + v_{n-2}\beta + ... + v_l\beta^{n-l-1},$$
$$R_n(\beta) = v_l\beta^{n-l},$$
$$v_n = \frac{1}{l}\left(\sum_{i=0}^{l-1}\varepsilon_{n-i}\right)^2,$$

and $\sigma_n^2 = \varepsilon_n^2$ for $n = 1 \dots l$.

For each fixed l, we maximize the likelihood function with respect to the other parameters. Thus, we obtain $\hat{\alpha}(l)$, $\hat{\beta}(l)$ and $\hat{\omega}(l)$ for $l = 1 \dots l_{\max}$. Then, we minimize AICC function to choose order $l \in [1, l_{\max}]$:

$$\text{AICC}(\hat{\alpha}(l), \hat{\beta}(l), \hat{\omega}(l), l) = -2\ln L(\hat{\alpha}(l), \hat{\beta}(l), \hat{\omega}(l), l) + \frac{2(l+3)N}{(N-l-4)}.$$

This function is an AICC function for ARMA(l+1,1) model. Note that the discrete-time model (4.3) is very similar to GARCH(l,1) model, the only difference is the presence of the cross-product terms in the equation for volatility. And as mentioned in [10], any GARCH(p,q) model can be considered as an ARMA(p+q,q) model. Therefore, it is reasonable to assume that AICC function for our model is similar to that for ARMA(l+1,1) model.

We search iteratively to find parameters that maximize the likelihood using a combination of direct search method and variable metric method, known as the Broyden-Fletcher-Goldfarb-Shanno variant of Davidon-Fletcher-Powell maximization algorithm (see attached Program 2 and [55]). Table 4 shows the results and performance of the algorithm applied to collections of daily observations of S&P500

index during 1990-1993.

The algorithm seems to be stable in almost all cases, except for the year 1992 where the maximum of likelihood function was achieved on the boundary of the feasible region, defined by the constraints and, therefore, cannot be accepted as a local extremum.

It is interesting to compare estimated parameters for different years. The annual pools of data showed little similarity, on the contrary to the results for 1992-93 and 1990-93, where the estimated parameter values were very close. This is a strong argument in favor of the results for larger datasets.

These results can be checked by looking at the autocorrelation structure of $\{\varepsilon_n\}$, i.e. correlation of series $\{\varepsilon_n\}$ and $\{\varepsilon_{n+k}\}$ for each lag $k \geq 1$ (see Table 5). Really, as the table shows, the highest (by absolute value) autocorrelation for $\{\varepsilon_n\}$ is at the lag 7, which indicates the consistency with results based on the ML-AICC method.

Another test of consistency of our results is to look at how our model for σ_n^2 removes autocorrelations in $\{\varepsilon_n^2\}$. For that purpose, we consider autocorrelations for $\{\varepsilon_n^2\}$ and $\{\varepsilon_n^2/\sigma_n^2\}$. There is an efficient way to check it by using Ljung-Box statistic for both series. Its value is defined by

$$N \sum_{k=1}^{15} \frac{N+2}{N-k}\, \phi_k^2 = 160.64,$$
$$N \sum_{k=1}^{15} \frac{N+2}{N-k}\, \theta_k^2 = 14.18,$$

where $N = 1006$ is the total number of observations, k is the index for lag and ϕ_k, θ_k are the autocorrelations of $\{\varepsilon_n^2\}$ and $\{\varepsilon_n^2/\sigma_n^2\}$ resp. For 15 lags in total, zero autocorrelation hypothesis can be rejected with 95% confidence when the Ljung-Box statistic is greater than 25.

From these values, we see that there is a strong evidence for autocorrelation in $\{\varepsilon_n^2\}$, since its Ljung-Box statistic is over 160. And for the $\{\varepsilon_n^2/\sigma_n^2\}$ series the Ljung-Box statistic is about 14, suggesting that the autocorrelation has been largely removed by our model (4.3) with parameters obtained by ML-AICC method.

Note that to validate our model (4.3) we can apply the estimation technique to other pools of S&P500 data (say, for years 1994-1997). We can use a new set of parameters to price options in comparison with market option prices, thereby confirming the model. This is a subject for future investigations.

4.3 Fitting option price data with CIR-GARCH model

Parameters of our stock price model (3.1) with σ defined by (3.12) are

$$V = 0.0141, \quad \alpha = 0.0575, \quad \gamma = 0.0539, \quad \tau = 0.028, \quad \sigma_0^2 = 0.0111.$$

These parameters were estimated from S&P500 index data for years 1990-1993 (see Section 4.2.2) using the maximum likelihood method. The only parameter that we could not estimate from the index data was the risk-free rate r. In this section, we use the S&P500 option trade data to derive this parameter and compare it with the U.S. treasury yield curve.

Given the option price as a function F of its unknown parameters θ, we can find the values of the parameters that fit the option price data. The fit can be achieved using the following least-squares optimization

$$\min_{\theta} \sum_{i=1}^{n} (F(\theta, T_i, K_i) - C_i)^2,$$

where $F(\theta, T_i, K_i)$ is the European call option price as a function of unknown parameters θ, time to maturity T_i and strike price K_i; C_i is the option price observed in the market.

The parameter r is the only unknown parameter. For any fixed r, T_i and K_i, the function $F(r, T_i, K_i)$ is given by the following expectation

$$F(r, T_i, K_i) = E\left[e^{-rT_i} \max(S(T_i) - K_i, 0)\right],$$

where $S(t)$ follows (3.1) with σ defined by (3.12). It can be computed using the accelerated Monte Carlo from Section 3.4.

In Figure 4 we present numerical results for the estimation of parameter r from the market data (solid lines) in comparison with the U.S. treasury yield curve (dashed lines). For longer maturities, the estimated risk-free rate agrees with the yield curve. However, there is some disagreement for shorter maturities. This shows that the model underprices the close-to-maturity options, however it prices well the options with longer maturities.

Observe that when the option maturity increases, the estimated risk-free rate decreases to some mean-reversion level. It is well-known that a bond spot rate is mean-reverting. Therefore, it is reasonable to assume that the risk-free rate implied in S&P500 options market possesses this property. Hence, it is of great interest to fit the rate r with some of the well-known models of the spot rate, e.g. Cox-Ingersoll-

Ross (CIR) model

$$dr(t) = a(m - r(t))\, dt + b\sqrt{r(t)}\, dW^r(t), \quad r(0) = r_0.$$

In this case, the stock price process follows

$$dS(t) = r(t)S(t)\, dt + \sigma(t)S(t)\, dW(t)$$

with $\sigma(t)$ same as in (3.12), where $W^r(t)$ and $W(t)$ are assumed uncorrelated.

Estimation of parameters m, a, b and r_0 is performed using the same least-squares approach. First, the European call option price is given by the following expectation

$$F(\theta, T_i, K_i) = E\left[\exp\left\{-\int_0^{T_i} r(u)du\right\}\max(S(T_i) - K_i, 0)\right],$$

where $\theta = (m, a, b, r_0)$. The corresponding option price estimator is

$$\frac{1}{N}\sum_{n=1}^{N}\exp\left\{-\int_0^{T_i} r^{(n)}(u)du\right\}\max(S^{(n)}(T_i) - K_i, 0).$$

Iteratively performing Monte Carlo simulations of the option price and comparing it with the market data, we obtain the following parameter estimates

$$m = 0.0075, \quad a = 8.5, \quad b = 0.3, \quad r_0 = 0.07.$$

In Figure 5, we show the fit of the simulated option prices with the prices observed in the market. Note that the fit is exact except for some out-of-money options.

Figure 6 presents the corresponding implied volatility plots.

5 Stochastic state-dependent delay differential equations

In this Chapter, we extend the models discussed in Section 2.3 to include a varying time delay which depends on values of the state, i.e. $S(t)$ and $\sigma(t)$. We prove the existence and uniqueness of the solution to a general stochastic state-dependent delay differential equation (SSDDE). We also prove convergence of the Euler discrete-time approximation scheme for SSDDEs and provide the order of convergence. Using this approximation result, we perform Monte Carlo simulations of the stock price process with state-dependent delay and show viability of the model through a variety of implied volatility plots.

5.1 Existence of a solution to SSDDE

Here, we shall establish the existence of a solution to the following multi-dimensional SSDDE:

$$\begin{cases} dX(t) = F(X(t), X(t-\tau))\,dt + G(X(t), X(t-\tau))\,dW(t), \\ X(t) = \varphi(t), \quad t \in [-\delta, 0], \end{cases} \tag{5.1}$$

where $\tau = \tau(X(t), X(t-\kappa))$, $0 < \delta_0 \leq \tau(s_1, s_2) \leq \delta$ for $s_1, s_2 \in \mathbf{R}^n$, $\kappa \in [\delta_0, \delta]$ and $\{W(t)\}$ is a Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

In what follows, $|\cdot|$ is the Euclidean norm. For any $a < b$, $L_2(\Omega, C[a,b])$ is the space of $C[a,b]$-valued random variables equipped with the norm defined by

$$\|\eta\|_L := \sqrt{E\|\eta\|_c^2}\,,$$

where $\|\eta\|_c = \sup_{a\le t\le b} |\eta(t)|$.

For $\alpha \in (0,1]$, we define a set $S_\alpha \subset L_2(\Omega, C[-\delta,0])$ by

$$S_\alpha := \left\{\eta \mid \exists M\ \forall t_1,t_2 \in [-\delta,0]:\ E|\eta(t_1)-\eta(t_2)|^2 \le M|t_1-t_2|^{2\alpha}\right\}.$$

Theorem 5.1 *Assume F, G and τ are continuous in their arguments. Then for any $\mathcal{F}_0$-measurable initial data $\varphi \in L_2(\Omega, C[-\delta,0])$ there exists a solution of SSDDE* (5.1) *defined on* $[0,+\infty)$.

Proof: We use the so-called method of steps to construct a solution to (5.1). Note that for any $t \in [n\delta_0,(n+1)\delta_0]$, $n \ge 0$ we have the following.

$$\begin{aligned} X(t) =& X(n\delta_0) + \int_{n\delta_0}^t F(X(s), X(s-\tau(X(s),X(s-\kappa))))\,ds \\ &+ \int_{n\delta_0}^t G(X(s), X(s-\tau(X(s),X(s-\kappa))))\,dW(s). \end{aligned} \tag{5.2}$$

Since $\tau(x,y) \ge \delta_0$ for $x,y \in \mathbf{R}^n$, we have $s-\tau(X(s),X(s-\kappa)) \le s-\delta_0$ and (5.2) becomes a stochastic ODE. Note that since $\{X(u), -\delta \le u \le s-\delta_0\}$ is a.s. continuous, there is an a.s. continuous solution to (5.2) defined on $[n\delta_0,(n+1)\delta_0]$ (see [58]).

5.2 Discrete-time approximations of SSDDEs

In this section, we prove that the Euler discrete-time scheme for a SSDDE with a special form has 1/2-strong order of convergence over $[0, T]$. This extends the result for stochastic ODEs. We also show that the scheme for a slightly more restrictive SSDDE has $1/2^{\lceil \frac{T}{\kappa}+1 \rceil}$-strong order of convergence, and we derive the uniqueness of solutions as a corollary.

5.2.1 SSDDE: type I

Consider the following special case of (5.1):

$$\begin{cases} dX_1(t) = f(X(t), X_2(t-\tau))\, dt + g(X(t), X_2(t-\tau))\, dW(t), \\ dX_2(t) = z(X(t), X_2(t-\tau))\, dt, \\ X(t) = [X_1(t), X_2(t)]^T = \varphi(t), \quad t \in [-\delta, 0], \end{cases} \tag{5.3}$$

where $\tau = \tau(X(t))$, $X_1(t) \in \mathbf{R}^{n_1}$, $X_2(t) \in \mathbf{R}^{n_2}$ and $X(t) \in \mathbf{R}^{n_1+n_2}$. Note that only the X_2-component has state-dependent delayed effect.

For a fixed $h > 0$ and $t \in \mathbf{R}$, we denote $\lfloor t \rfloor = h[t/h]$, where $[\cdot]$ is the integer part. *Strong Euler approximation scheme* for (5.3) is defined as follows:

$$\begin{cases} d\bar{X}_1(t) = f(\bar{X}(\lfloor t \rfloor), \bar{X}_2(\lfloor t \rfloor - \lfloor \bar{\tau} \rfloor))\, dt + g(\bar{X}(\lfloor t \rfloor), \bar{X}_2(\lfloor t \rfloor - \lfloor \bar{\tau} \rfloor))\, dW(t), \\ d\bar{X}_2(t) = z(\bar{X}(\lfloor t \rfloor), \bar{X}_2(\lfloor t \rfloor - \lfloor \bar{\tau} \rfloor))\, dt, \\ \bar{X}(t) = [\bar{X}_1(t), \bar{X}_2(t)]^T = \varphi(t), \quad t \in [-\delta, 0], \end{cases} \tag{5.4}$$

where $\lfloor \bar{\tau} \rfloor = \lfloor \tau(\bar{X}(\lfloor t \rfloor)) \rfloor$.

Theorem 5.2 *Assume that f, g, z and τ are Lipschitz continuous with respect to all of their arguments. Then for any $\mathcal{F}_0$-measurable initial data $\varphi \in S_1$, there exists a constant $C(T,\varphi)$ such that for any $\alpha \in [0,1)$*

$$E\left[\sup_{t\in[0,T]} |X(t)-\bar{X}(t)|^2\right] \le C(T,\varphi)\ h^{\alpha}$$

for a sufficiently small h, where h is the partition's mesh size, X and $\bar{X}$ satisfy (5.3) *and* (5.4) *respectively. Moreover, the solution X of* (5.3) *is pathwise unique.*

Proof: Using representations (5.3) and (5.4) for X and $\bar{X}$, and Doob's inequality, we get

$$\begin{aligned}
&\|X-\bar{X}\|^2_{L_2(\Omega,C[0,t])}\\
=&E\left[\sup_{u\in[0,t]} |X(u)-\bar{X}(u)|^2\right]\\
\le&2E\left[\sup_{u\in[0,t]} |\int_0^u (f(X(s),X_2(s-\tau))-f(\bar{X}(\lfloor s\rfloor),\bar{X}_2(\lfloor s\rfloor-\lfloor\bar{\tau}\rfloor)))ds|^2\right]\\
&+2E\left[\sup_{u\in[0,t]} |\int_0^u (g(X(s),X_2(s-\tau))-g(\bar{X}(\lfloor s\rfloor),\bar{X}_2(\lfloor s\rfloor-\lfloor\bar{\tau}\rfloor)))dW(s)|^2\right]\\
&+E\left[\sup_{u\in[0,t]} |\int_0^u (z(X(s),X_2(s-\tau))-z(\bar{X}(\lfloor s\rfloor),\bar{X}_2(\lfloor s\rfloor-\lfloor\bar{\tau}\rfloor)))ds|^2\right]\\
\le&2t\int_0^t E|f(X(s),X_2(s-\tau))-f(\bar{X}(\lfloor s\rfloor),\bar{X}_2(\lfloor s\rfloor-\lfloor\bar{\tau}\rfloor))|^2 ds\\
&+8\int_0^t E|g(X(s),X_2(s-\tau))-g(\bar{X}(\lfloor s\rfloor),\bar{X}_2(\lfloor s\rfloor-\lfloor\bar{\tau}\rfloor))|^2 ds\\
&+t\int_0^t E|z(X(s),X_2(s-\tau))-z(\bar{X}(\lfloor s\rfloor),\bar{X}_2(\lfloor s\rfloor-\lfloor\bar{\tau}\rfloor))|^2 ds.
\end{aligned} \tag{5.5}$$

We now estimate each term in (5.5). First of all, we have

$$\int_0^t E|f(X(s), X_2(s-\tau)) - f(\bar{X}(\lfloor s\rfloor), \bar{X}_2(\lfloor s\rfloor - \lfloor\bar{\tau}\rfloor))|^2 ds \\ \le 5\left[J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t)\right],$$

where

$$\begin{aligned}
J_1(t) &= \int_0^t E|f(X(s), X_2(s-\tau)) - f(X(\lfloor s\rfloor), X_2(s-\tau))|^2\, ds,\\
J_2(t) &= \int_0^t E|f(X(\lfloor s\rfloor), X_2(s-\tau)) - f(\bar{X}(\lfloor s\rfloor), X_2(s-\tau))|^2\, ds,\\
J_3(t) &= \int_0^t E|f(\bar{X}(\lfloor s\rfloor), X_2(s-\tau)) - f(\bar{X}(\lfloor s\rfloor), X_2(\lfloor s\rfloor - \lfloor\tau\rfloor))|^2\, ds,\\
J_4(t) &= \int_0^t E|f(\bar{X}(\lfloor s\rfloor), X_2(\lfloor s\rfloor - \lfloor\tau\rfloor)) - f(\bar{X}(\lfloor s\rfloor), X_2(\lfloor s\rfloor - \lfloor\bar{\tau}\rfloor))|^2\, ds,\\
J_5(t) &= \int_0^t E|f(\bar{X}(\lfloor s\rfloor), X_2(\lfloor s\rfloor - \lfloor\bar{\tau}\rfloor)) - f(\bar{X}(\lfloor s\rfloor), \bar{X}_2(\lfloor s\rfloor - \lfloor\bar{\tau}\rfloor))|^2\, ds.
\end{aligned}$$

Here, $\tau = \tau(X(s))$, $\lfloor\tau\rfloor = \lfloor\tau(X(\lfloor s\rfloor)\rfloor$ and $\lfloor\bar{\tau}\rfloor = \lfloor\tau(\bar{X}(\lfloor s\rfloor)\rfloor$. Since f and τ are Lipschitz continuous and X is pathwise $\alpha/2$-Lipschitz continuous for any $\alpha \in [0,1)$ with the constant $M(\varphi)$, a.s. (follows from the law of iterated logarithm, see [60]), we obtain

$$
\begin{aligned}
J_1(t) &\le L\int_0^t E|X(s)-X(\lfloor s\rfloor)|^2\,ds \le L\int_0^t M(\varphi)(s-\lfloor s\rfloor)^\alpha\,ds\\
&\le LtM(\varphi)\,h^\alpha \equiv C_1(t,\varphi)\,h^\alpha,\\
J_2(t) &\le L\int_0^t E|X(\lfloor s\rfloor)-\bar X(\lfloor s\rfloor)|^2\,ds \le L\int_0^t \|X-\bar X\|^2_{L_2(\Omega,C[0,s])}\,ds,\\
J_3(t) &\le L\int_0^t E|X_2(s-\tau)-X_2(\lfloor s\rfloor-\lfloor\tau\rfloor)|^2\,ds\\
&\le LM_2(\varphi)\int_0^t E|s-\tau-\lfloor s\rfloor+\lfloor\tau\rfloor|^2\,ds\\
&\le 2LM_2(\varphi)\int_0^t (s-\lfloor s\rfloor)^2\,ds + 2LM_2(\varphi)\int_0^t E(\tau-\lfloor\tau\rfloor)^2\,ds\\
&\le 6LM_2(\varphi)t\,h^2 + 4LM_2(\varphi)L_\tau\int_0^t E|X(s)-X(\lfloor s\rfloor)|^2\,ds\\
&\le 6LM_2(\varphi)t\,h^2 + 4LM_2(\varphi)L_\tau M(\varphi)t\,h^\alpha \equiv C_2(t,\varphi)\,h^2 + C_3(t,\varphi)\,h^\alpha.
\end{aligned}
$$

Here, $\sqrt{L}$ and $\sqrt{L_\tau}$ are the Lipschitz constants of f and τ, respectively. We also used the fact that φ and X_2 are Lipschitz continuous with constant $M_2(\varphi)$ because the diffusion coefficient of second equation in (5.3) is zero. We also have

$$
\begin{aligned}
J_4(t) &\le LM_2(\varphi)\int_0^t E|\lfloor\tau\rfloor-\lfloor\bar\tau\rfloor|^2\,ds\\
&\le 2LM_2(\varphi)t\,h^2 + 2LM_2(\varphi)L_\tau\int_0^t E|X(\lfloor s\rfloor)-\bar X(\lfloor s\rfloor)|^2\,ds\\
&\le 2LM_2(\varphi)t\,h^2 + 2LM_2(\varphi)L_\tau\int_0^t \|X-\bar X\|^2_{L_2(\Omega,C[0,s])}\,ds\\
&\equiv C_4(t,\varphi)\,h^2 + C_5(\varphi)\int_0^t \|X-\bar X\|^2_{L_2(\Omega,C[0,s])}\,ds,\\
J_5(t) &\le L\int_0^t E|X_2(\lfloor s\rfloor-\lfloor\bar\tau\rfloor)-\bar X_2(\lfloor s\rfloor-\lfloor\bar\tau\rfloor)|^2\,ds\\
&\le L\int_0^t \|X-\bar X\|^2_{L_2(\Omega,C[0,s])}\,ds.
\end{aligned}
$$

Therefore,

$$\int_0^t E|f(X(s), X_2(s-\tau)) - f(\bar{X}(\lfloor s \rfloor), \bar{X}_2(\lfloor s \rfloor - \lfloor \bar{\tau} \rfloor))|^2 ds$$
$$\leq C_6(t,\varphi)\ h^\alpha + C_7(\varphi) \int_0^t \|X - \bar{X}\|^2_{L_2(\Omega, C[0,s])}\ ds$$

for a sufficiently small h. Carrying out the same analysis for terms in (5.5) with g and z, we obtain

$$\|X - \bar{X}\|^2_{L_2(\Omega, C[0,t])} \leq A(\varphi, T) \int_0^t \|X - \bar{X}\|^2_{L_2(\Omega, C[0,s])}\ ds + B(T, \varphi)\ h^\alpha \tag{5.6}$$

for certain constants A and B. Consequently, an application of the Grownwall inequality yields

$$\|X - \bar{X}\|^2_{L_2(\Omega, C[0,T])} \leq B(T, \varphi) e^{A(\varphi, T)T}\ h^\alpha.$$

Uniqueness of the solution follows from this inequality, and the theorem is proved.

5.2.2 SSDDE: type II

In this subsection, we consider SSDDE of the following more general form:

$$\begin{cases} dX(t) = F(X(t), X(t-\tau))\ dt + G(X(t), X(t-\tau))\ dW(t), \\ X(t) = \varphi(t), \quad t \in [-\delta, 0], \end{cases} \tag{5.7}$$

where $\tau = \tau(X(t-\kappa))$. Note that τ depends on κ-delayed value of X only. The Euler discrete-time scheme for (5.7) is given by

$$\begin{cases} d\bar{X}(t) = F(\bar{X}(\lfloor t \rfloor), \bar{X}(\lfloor t \rfloor - \lfloor \bar{\tau} \rfloor))\, dt + G(\bar{X}(\lfloor t \rfloor), \bar{X}(\lfloor t \rfloor - \lfloor \bar{\tau} \rfloor))\, dW(t), \\ \bar{X}(t) = \varphi(t), \quad t \in [-\delta, 0], \end{cases} \tag{5.8}$$

where $\lfloor \bar{\tau} \rfloor = \lfloor \tau(\bar{X}(\lfloor t-\kappa \rfloor)) \rfloor$.

Theorem 5.3 *Assume F, G and τ are Lipschitz continuous with respect to all of their arguments. Then for any $\mathcal{F}_0$-measurable initial data $\varphi \in S_{1/2}$, there exists a constant $C(T,\varphi)$ such that for any $\alpha \in [0,1)$*

$$E\left[\sup_{t\in[0,T]} |X(t) - \bar{X}(t)|^2\right] \leq C(T,\varphi)\, h^n$$

for a sufficiently small h, where h is the partition's mesh size, $n = \alpha(\alpha/2)^{\lceil \frac{T}{\kappa} \rceil}$ and $\lceil \cdot \rceil$ is the ceiling integer function. Moreover, the solution X of (5.7) is pathwise unique.

Proof: We use similar arguments used in the proof of Theorem 5.2, with necessary modifications. Since X is pathwise $\alpha/2$-Lipschitz continuous for any $\alpha \in [0,1)$, a.s., estimation for $J_3(t)$ and $J_4(t)$ becomes

$$\begin{aligned} J_4(t) &\le LM(\varphi) \int_0^t E|\lfloor\tau\rfloor - \lfloor\bar{\tau}\rfloor|^\alpha \, ds \\ &\le LM(\varphi)t\ h^\alpha + LM(\varphi)L_\tau^\alpha \int_0^t E|X(\lfloor s-\kappa\rfloor) - \bar{X}(\lfloor s-\kappa\rfloor)|^\alpha \, ds \\ &\le LM(\varphi)t\ h^\alpha + LM(\varphi)L_\tau^\alpha \int_0^t \|X - \bar{X}\|^\alpha_{L_2(\Omega, C[-\delta, s-\kappa])} \, ds \\ &\equiv C_4(t,\varphi)\ h^\alpha + C_5(\varphi) \int_0^t \|X - \bar{X}\|^\alpha_{L_2(\Omega, C[-\delta, s-\kappa])} \, ds, \end{aligned}$$

and

$$J_3(t) \le C_2(t,\varphi)h^\alpha + C_3(t,\varphi)h^{\alpha^2/2}.$$

In addition, we should replace (5.6) by

$$\varepsilon(t) \le A(\varphi, T) \int_0^t \big(\varepsilon(s) + \varepsilon^{\alpha/2}(s-\kappa)\big)\ ds + B(T,\varphi)\ h^{\alpha^2/2},$$

where $\varepsilon(t) = \|X - \bar{X}\|^2_{L_2(\Omega, C[-\delta, t])}$. Since X and $\bar{X}$ have the same initial data, $\varepsilon(t) = 0$ for $t \le 0$. By Grownwall inequality, we then obtain

$$\varepsilon(t) \le Be^{A\kappa}h^{\alpha^2/2} \quad \text{for } t \in [0, \kappa],$$

$$\varepsilon(t) \le A \int_0^t \varepsilon(s)\ ds + B_2 h^{\alpha^3/4} \quad \text{for } t \in [\kappa, 2\kappa],$$

$$\varepsilon(t) \le B_2 e^{2A\kappa} h^{\alpha^3/4} \quad \text{for } t \in [\kappa, 2\kappa],$$

where $B_2 = Bh_0^{\alpha^2/2 - \alpha^3/4} + AB^{\alpha/2}e^{A\kappa\alpha/2}\kappa$. By iterations, we obtain

$$\varepsilon(t) \le B_n e^{nA\kappa}\ h^{\alpha(\alpha/2)^n} \quad \text{for } t \in [(n-1)\kappa, n\kappa]$$

where $B_n = Bh_0^{\alpha^2/2-\alpha^{n+1}/2^n} + AB_{n-1}^{\alpha/2}e^{A(n-1)\kappa\alpha/2}(n-1)\kappa$ for $n \geq 2$ and $B_1 = B$. This completes the proof.

5.3 Continuous-time GARCH model with state-dependent delay

The following model for stock price x and its volatility $\sqrt{y}$ was derived in Section 2.3:

$$
\begin{aligned}
dx(t) &= rx(t)\,dt + \sqrt{y(t)}x(t)\,dW(t), \\
\frac{dy(t)}{dt} &= \gamma V + \frac{\alpha}{\tau}\left\{\ln\frac{x(t)}{x(t-\tau)} - \mu\tau + \frac{1}{2}\int_{t-\tau}^{t} y(s)\,ds\right\}^2 - (\alpha+\gamma)y(t),
\end{aligned}
\tag{5.9}
$$

where $\tau > 0$ is a constant. The model is derived from discrete-time GARCH(1,1) model, and parameter estimation for S&P500 from Chapter 4 shows that the delay parameter varies considerably from year to year. This leads us to the assumption that τ is a function of state values. In this section, we assume $\tau = \tau(y(t-\kappa))$ so a local Lipschitz version of Theorem 5.3 can be applied.

From the previous subsection, the Euler discrete-time scheme given below is convergent to the unique solution. The scheme is given by

$$
\begin{aligned}
x_{n+1} - x_n &= rx_n\Delta t + \sqrt{y_n}x_n\sqrt{\Delta t}\varepsilon_n, \\
\frac{y_{n+1}-y_n}{\Delta t} &= \gamma V + \frac{\alpha}{\tau(y_{n-k})}\left\{\ln\frac{x_n}{x_{n-N(y_{n-k})}} - \mu\tau(y_{n-k}) + \frac{\Delta t}{2}\sum_{i=0}^{N(y_{n-k})} y_{n-i}\right\}^2 \\
&\quad - (\alpha+\gamma)y_n,
\end{aligned}
\tag{5.10}
$$

where $\tau(y_{n-k}) = \delta_0 + \hat{\tau}\exp(\rho y_{n-k})$ for some $-\rho, \hat{\tau}, \delta_0 > 0$, $N = [(\hat{\tau} + \delta_0)/\Delta t]$,

$N(y_{n-k}) = [\tau(y_{n-k})/\Delta t]$, $k = [\kappa/\Delta t]$, $[\cdot]$ is the integer part and $\{\varepsilon_n\}_{n\geq 0}$ are i.i.d. Normal(0,1). Here, the initial data (x_n, y_n) are provided for $n = -N, \ldots, 0$.

The particular choice of function τ is due to the following empirical observation: since y represents volatility of the stock, for large y, the prices are more volatile, and therefore, trading is more active. Assuming the market's response to changes in the stock price is faster when the volatility is higher, we then conclude that the delay is a decreasing function of the volatility.

Let us try to find a fair price for the European call option written on the stock with maturity T and strike price K. It is known that the option price C is given by the following expectation:

$$C = E\left[e^{-rT}\max(x_T - K, 0)\right],$$

where r is risk-free spot rate and x_T is stock price at the time T. This expectation can be found using a Monte Carlo simulation of x_T approximated by the scheme (5.10).

Some simulation results are provided in the attached figures for different functions of state-dependent delay τ. They are presented as plots of implied volatility against strike price K. Note that implied volatility is computed using the inverse of Black-Scholes formula applied to simulated option price C.

It is well-known that the curve of the implied volatility of *market* option price has a U-shape, this is further confirmed by our plots (see Figures 7-10). Observe also that the curvature of the graph is getting larger and larger when the value of the delay τ is increased. A constant delay cannot be used to control the height of the curve independently of the curvature, whereas varying delay can. Moreover, we provide some plots by using τ as an increasing or a periodic function to illustrate the variety of curves we can obtain. Solid lines represent 95%-confidence bounds for 10^6 simulations and dashed lines represent 95%-confidence bounds for 10^7 simulations.

6 Appendices

6.1 Derivation of continuous-time analogue of GARCH

The discrete-time model has the following form

$$
\begin{aligned}
Y_n &= Y_{n-1} + \left(\mu - \frac{1}{2}\sigma_n^2\right) + \sigma_n \varepsilon_n, \\
\sigma_n^2 &= \gamma V + \frac{\alpha}{l}\left(\sum_{k=1}^{l} \sigma_{n-k}\, \varepsilon_{n-k}\right)^2 + (1-\alpha-\gamma)\sigma_{n-1}^2, \\
\{\varepsilon_n\}_{n\geq 1} &\sim i.i.d.\ N(0,1),
\end{aligned}
\tag{6.1}
$$

where initial data are given by $(Y_i, \sigma_i^2) = (y_i, v_i)$ with $i = -l \ldots 0$. For any fixed $l \geq 1$ define a partition $\pi = \{nh \mid n \geq -l,\ h = \tau/l\}$. Then discrete-time model (6.1) defined over π takes the form

$$
\begin{aligned}
Y_{nh}^{\pi} &= Y_{(n-1)h}^{\pi} + \left(\mu - \frac{1}{2}(\sigma_{nh}^{\pi})^2\right) h + \sigma_{nh}^{\pi}\, \varepsilon_{nh}^{\pi}, \\
(\sigma_{nh}^{\pi})^2 &= \gamma^{\pi} V + \frac{\alpha^{\pi}}{l}\left(\sum_{k=1}^{l} \sigma_{(n-k)h}^{\pi}\, h^{-\frac{1}{2}}\, \varepsilon_{(n-k)h}^{\pi}\right)^2 + (1-\alpha^{\pi}-\gamma^{\pi})\, (\sigma_{(n-1)h}^{\pi})^2, \\
\{\varepsilon_{nh}^{\pi}\}_{n\geq 1} &\sim i.i.d.\ N(0,h),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
Y^{\pi}_{nh} =& Y^{\pi}_{(n-1)h} + \left(\mu - \frac{1}{2}(\sigma^{\pi}_{nh})^2\right) h + \sigma^{\pi}_{nh}\, \varepsilon^{\pi}_{nh},\\
(\sigma^{\pi}_{nh})^2 =& (\sigma^{\pi}_{(n-1)h})^2 + \gamma^{\pi} V +\\
&+ \frac{\alpha^{\pi}}{\tau}\left(Y^{\pi}_{(n-1)h} - Y^{\pi}_{(n-l-1)h} - \sum_{k=1}^{l}\left(\mu - \frac{1}{2}(\sigma^{\pi}_{(n-k)h})^2\right) h\right)^2 -\\
&- (\alpha^{\pi} + \gamma^{\pi})\,(\sigma^{\pi}_{(n-1)h})^2.
\end{aligned}
$$

Let us take $\gamma^{\pi} = \gamma h$, $\alpha^{\pi} = \alpha h$ and define $(Y^{\pi}(t), \sigma^{\pi}(t))$ by

$$
\begin{aligned}
&Y^{\pi}(t) = Y^{\pi}_{(n-1)h} + \left(\mu - \frac{1}{2}(\sigma^{\pi}_{nh})^2\right)(t-(n-1)h) + \sigma^{\pi}_{nh}\left(W(t) - W((n-1)h)\right),\\
&(\sigma^{\pi}(t))^2 = (\sigma^{\pi}_{(n-1)h})^2 + [\gamma V +\\
&+ \frac{\alpha}{\tau}\left(Y^{\pi}_{(n-1)h} - Y^{\pi}_{(n-l-1)h} - \sum_{k=1}^{l}\left(\mu - \frac{1}{2}(\sigma^{\pi}_{(n-k)h})^2\right) h\right)^2 - (\alpha+\gamma)\,(\sigma^{\pi}_{(n-1)h})^2]\\
&\times (t-(n-1)h)
\end{aligned}
$$

for $(n-1)h \le t < nh$ with $n \ge 1$, where $W(t)$ is a Wiener process defined on our probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathcal{P})$. Notice that $(Y^{\pi}(t), \sigma^{\pi}(t))$ is a continuous mapping from $[-\tau, \infty) \times \Omega$ to R^2 and its values coincide with $(Y^{\pi}_{nh}, \sigma^{\pi}_{nh})$ for $t = nh$ with $n \ge 0$. We define $v^{\pi}(t) = v_i + (v_{i+1} - v_i)(t - ih)h^{-1}$ and $y^{\pi}(t) = y_i + (y_{i+1} - y_i)(t - ih)h^{-1}$ for $ih \le t < (i+1)h$ with $i = -l, \cdots - l + 1$.

Let us consider a SDDE

$$dY(t) = (\mu - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dW(t),$$
$$\frac{d\sigma^2(t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left(Y(t) - Y(t-\tau) - \int_{t-\tau}^{t}(\mu - \frac{1}{2}\sigma^2(u))du\right)^2 - (\alpha+\gamma)\sigma^2(t), \tag{6.2}$$

with the initial data given by $(Y(t), \sigma^2(t)) = (y(t), v(t))$ for $t \in [-\tau, 0]$. By defining $S(t) = \exp(Y(t))$ with $\varphi(t) = \exp(y(t))$ and applying the Ito's lemma, we conclude that $S(t)$ coincides with the process introduced in Section 2.3.

Now if the initial data of (6.1) and (6.2) are close in the sense that

$$||y^\pi - y||^2 + ||v^\pi - v||^2 \leq Ch \tag{6.3}$$

for some constant $C > 0$ then $(Y^\pi(t), (\sigma^\pi(t))^2)$ and $(Y(t), \sigma^2(t))$ are close in the following sense (see [34])

$$E\int_0^T |Y^\pi(t) - Y(t)|^2 dt + E\int_0^T |(\sigma^\pi(t))^2 - \sigma^2(t)|^2 dt < C'h$$

for some constant $C' > 0$, under the regularity conditions for coefficients of (6.2). Namely,

$$|G(0)| + |H(0)| < \infty,$$

$$|G(\eta) - G(\xi)| + |H(\eta) - H(\xi)| \leq L||\eta - \xi||, \tag{6.4}$$

for some $L > 0$ and for all $\eta, \xi \in C([-\tau, 0], R^2)$, where

$$H(\eta) = \begin{bmatrix} \mu - \eta_2(0)/2 \\ \gamma V + (\alpha/\tau)\left(\eta_1(0) - \eta_1(-\tau) - \int_{-\tau}^{0}(\mu - \eta_2(\theta)/2)d\theta\right)^2 - (\alpha+\gamma)\eta_2(0) \end{bmatrix},$$

$$G(\eta) = \begin{bmatrix} \sqrt{\eta_2(0)} & 0 \\ 0 & 0 \end{bmatrix}$$

and $|\cdot|$, $\|\cdot\|$ are Euclidean norm and supremum norm, respectively. Note that the convergence result still holds when condition (6.4) is satisfied locally in $C([-\tau, 0], R^2)$.

In other words, by choosing continuous functions $y(t)$ and $v(t)$ such that (6.3) is satisfied for the partition π defined by every small $h > 0$ we ensure the convergence of the solution of discrete-time model (6.1) to the solution of continuous-time model (6.2) in the L^2-norm as h tends to zero.

6.2 Proof of Lemma 3.1

Here, we give a sketched proof of Lemma 3.1. Fix $t > 0$ and denote $C \ni x = S_t$ with $S(t)$ satisfying (3.1). Then for a sufficiently small s

$$[F(t+s, x_s) - F(t, x)] = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \int_{-\tau}^{0} h(\theta - s)\left[H(x(\theta), x(s), t+s) - H(x(\theta), x(s), t)\right] d\theta,$$
$$I_2 = \int_{-\tau}^{0} (h(\theta - s) - h(\theta)) H(x(\theta), x(s), t) d\theta,$$
$$I_3 = \int_{-\tau}^{0} h(\theta)\left[H(x(\theta), x(s), t) - H(x(\theta), x(0), t)\right] d\theta,$$
$$I_4 = \int_{0}^{s} h(\theta - s) H(x(\theta), x(s), t+s) d\theta,$$
$$I_5 = -\int_{-\tau}^{-\tau+s} h(\theta - s) H(x(\theta), x(s), t+s) d\theta.$$

Then, by letting $s \to 0$,

$$I_1 \to \int_{-\tau}^{0} h(\theta) H_3'(x(\theta), x(0), t) d\theta dt,$$
$$I_2 \to -\int_{-\tau}^{0} h'(\theta) H(x(\theta), x(0), t) d\theta dt,$$
$$I_3 \to \int_{-\tau}^{0} h(\theta) TH(x(\theta), x(0), t) d\theta dt + \int_{-\tau}^{0} h(\theta)\sigma(t, x) x(0) H_2'(x(\theta), x(0), t) d\theta \; dW(t),$$
$$I_4 \to \quad h(0) H(x(0), x(0), t) dt,$$
$$I_5 \to -\; h(-\tau) H(x(-\tau), x(0), t) dt,$$

where

$$TH(x(\theta), x(0), t) = r x(0) H_2'(x(\theta), x(0), t) + \frac{\sigma^2(t, x) x^2(0)}{2} H_{22}''(x(\theta), x(0), t).$$

The limit for I_3 is obtained by using Ito's lemma. Then expression (3.5) follows.

7 Conclusions and future work

In this work, we consider the problem of option pricing in the security market where volatility depends on time and delayed values of the stock. The thesis involves solving an integral partial differential equation for a function which defines the option price. We develop a numerical scheme to solve this equation in the general form. We also provide an alternative way to price options through accelerated Monte Carlo simulations.

In order to have a viable model for the volatility, we derive a continuous-time equivalent of GARCH(1,1)-model for stochastic volatility with delay. The model assumes the form of a system of SDDEs. We apply the general option pricing technique to this general model and compare the results via our numerical scheme with those using Monte Carlo simulations.

We also address the important issue of parameter estimation. The time delay estimation results based on market data show that the parameter varies from year to year. This led to a model with state-dependent delay. The model takes the form of a system of stochastic state-dependent delay differential equations (SSDDEs). Some basic results, such as the existence and uniqueness of the solution to a SSDDE, and the convergence of discrete-time approximations of SSDDEs, are derived. The approximation result is used to justify the convergence of a simulated discrete-time scheme. The simulation results produce a variety of U-shaped implied volatility plots. This indicates the importance of studying models with state-dependent delay.

In our future work, we are planning to continue to derive and study models with delay that arise in option pricing. We are particularly interested in developing a general option pricing approach for models with state-dependent delay. A challenging problem is the estimation and selection of the functional relation of the time delay to the past stock price, and this requires close examination of market data and some new ideas and methods. In addition, it is of an interest to model options that are written on several underlying stocks. This would involve an extension of the current framework to model covariances of the stocks that follow a multivariate GARCH model.

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Program 1: Finite-difference method described in Section 3.3.

```
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include <stddef.h>
#define THE_END 1
#define FREE_ARG char*

float *vector(long nl, long nh) {
  float *v;
  v=(float *)malloc((size_t) ((nh-nl+1+THE_END)*sizeof(float)));
  if (!v) { printf("allocation failure in vector()"); exit(1); }
  return v-nl+THE_END;
}

float **matrix(long nrl, long nrh, long ncl, long nch) {
  long i, nrow=nrh-nrl+1, ncol=nch-ncl+1;
  float **m;
  m=(float **) malloc((size_t)((nrow+THE_END)*sizeof(float*)));
  if (!m) { printf("allocation failure 1 in matrix()"); exit(1); }
  m += THE_END;
  m -= nrl;
  m[nrl]=(float *) malloc((size_t)((nrow*ncol+THE_END)*sizeof(float)));
  if (!m[nrl]) { printf("allocation failure 2 in matrix()"); exit(1); }
  m[nrl] += THE_END;
  m[nrl] -= ncl;
  for (i=nrl+1; i<=nrh; i++) m[i]=m[i-1]+ncol;
  return m;
}

void free_vector(float *v, long nl, long nh) {
  free((FREE_ARG) (v+nl-THE_END));
}

void free_matrix(float **m, long nrl, long nrh, long ncl, long nch) {
  free((FREE_ARG) (m[nrl]+ncl-THE_END));
  free((FREE_ARG) (m+nrl-THE_END));
}

const float tau=0.028, r=0.0362, mu=0.0362, alpha=0.0575*250,
```

```
g=0.0539*250, sig0=0.0107, V=0.0141, Vb=0.0141;
const double Pi=3.1415926;

float sigma_const(float t, float x, float y) {
  return sig0;
}

float sigma_init(float t, float x, float y) {
  float temp;
  temp = log(x/y);
  return sig0*exp(-(alpha+g)*t)+(g*V+(alpha/tau)*temp*temp)*
  (1-exp(-(alpha+g)*t))/(alpha+g);
}

double NF(double z) {
  double zz, tailprob, t;
  zz=fabs(z)/sqrt(2.0);
  t=1.0/(1.0+0.5*zz);
  tailprob=0.5*t*exp(-zz*zz-1.26551223+t*(1.00002368+t*(0.37409196+
     t*(0.09678418+t*(-0.18628806+t*(0.27886807+t*(-1.13520398+
     t*(1.48851587+t*(-0.82215223+t*0.17087277)))))))));
  if (z<=0.0) return tailprob;
  else return 1.0-tailprob;
}

float bscall(float x, float t, float T, float K, float sigma) {
  float d1,d2,avsig;
  avsig = sigma*(T-t);
  d1 = (log(x/K)+(avsig/2)+r*(T-t))/sqrt(avsig);
  d2 = d1-sqrt(avsig);
  return x*NF(d1)-K*exp(-r*(T-t))*NF(d2);
}

float Fmax(float a, float b) {
  if (a>b) return a; else return b;
}

void rangephi(float phi[], int sizephi, int *low, int *up, float dS) {
  float templow=1000.0, tempup=0.0;
  for (int i=1; i<=sizephi; i++) {
    if (templow > phi[i]) templow=phi[i];
    if (tempup < phi[i]) tempup=phi[i];
  }
```

```
  *low = -(int)floor((templow-phi[sizephi])/dS);
  *up = (int)floor((tempup-phi[sizephi])/dS);
}

// Computation of H_ij

void fcal(float *f, int nlow, int nup, int dim, float x, float t, float T,
  float K, float dS, float dt, float (*sigma)(float, float, float)) {

  float **F,**Fold,xx,yy,ti,sum,tauh,ertau,x0,y0,A,B,C,D,f1,f2,fr,*bsp;
  int i,j,k,xdim,ydim;
  xdim = (2*dim+1)*(nlow+nup)+1;
  ydim = (2*dim+1)*(nlow+nup)+1;
  F = matrix(1,xdim,1,ydim);
  Fold = matrix(1,xdim,1,ydim);
  x0 = x-(nlow+dim*(nlow+nup))*dS;
  y0 = x-(nlow+dim*(nlow+nup))*dS;
  ertau = exp(r*tau);
  tauh = (ertau-1)/r;
  bsp = vector(1,xdim);

  for (i=1; i<=xdim; i++)
    for (j=1; j<=ydim; j++)
      Fold[i][j] = (x0-y0+(i-j)*dS)*Fmax(x0+(i-1)*dS-K,0);

  for (ti=T-dt; ti>=t; ti-=dt) {
    for (i=2; i<=xdim-1; i++)
      for (j=2; j<=ydim-1; j++)
        if (i!=j) {
        xx = x0+(i-1)*dS;
        yy = y0+(j-1)*dS;
        A = (*sigma)(ti,xx,yy)*xx*xx*dt/(2*dS*dS);
        B = r*xx*dt/(2*dS);
        D = ((xx-yy)/tauh)*(dt/(2*dS));
        C = D*ertau;
        F[i][j] = (1-2*A)*Fold[i][j]+C*(Fold[i][j+1]-Fold[i][j-1])+
          (B+A)*Fold[i+1][j]+(-B+A)*Fold[i-1][j]+(-D+B+A)*Fold[i][i+1]+
          (D-B+A)*Fold[i][i-1]-(A/2)*(Fold[i+1][i-1]+Fold[i-1][i+1]);
        }
        for (i=1; i<=xdim; i++) F[i][i] = 0.0;
        for (i=1; i<=xdim; i++) bsp[i] = bscall(x0+(i-1)*dS,ti,T,K,Vb);
        for (i=1; i<=xdim; i++) {
          F[i][1] = (x0-y0+(i-1)*dS)*bsp[i];
```

```
          F[i][ydim] = (x0-y0+(i-ydim)*dS)*bsp[i];
        }
        for (j=1; j<=ydim; j++) {
        F[1][j] = (x0-y0+(1-j)*dS)*bsp[1];
        F[xdim][j] = (x0-y0+(xdim-j)*dS)*bsp[xdim];
        }
        for (i=1; i<=xdim; i++)
          for (j=1; j<=ydim; j++)
            Fold[i][j] = F[i][j];
        }
        printf("\n");
        for (j=1; j<=nlow+nup+3; j++) {
          f[j] = F[nlow+dim*(nlow+nup)+1][j+dim*(nlow+nup)-1]/tauh;
          printf("\n%f",f[j]);
        }
        free_matrix(F,1,xdim,1,ydim);
        free_matrix(Fold,1,xdim,1,ydim);
        free_vector(bsp,1,xdim);
}

float opprice(float t, float T, float K, float phi[], int sizephi,
  float dS, float dt, int dim, float (*sigma)(float, float, float)) {

  float step,sum=0.0,*f;
  int nlow,nup,i;
  step=tau/(sizephi-1);
  rangephi(phi,sizephi,&nlow,&nup,dS);
  f=vector(1,nlow+nup+3);
  fcal(f,nlow,nup,dim,phi[sizephi],t,T,K,dS,dt,sigma);
  for (int k=2; k<=sizephi; k++) {
    i = (int)floor((phi[sizephi-k+1]-phi[sizephi])/dS)+nlow+2;
    sum += exp(r*(k-1)*step)*(f[i-1]-f[i+1])*step/(2*dS);
  }
  free_vector(f,1,nlow+nup+3);
  return sum;
}

int main(int argc, char *argv[]) {
  float *phi,opp,K,T,dS,dt;
  int opt,dim,j;
  phi=vector(1,8);
  dS=atof(argv[1]);  // 0.5
  dt=atof(argv[2]);  // 0.00001
```

```
  dim=atoi(argv[3]); // 5

  phi[1]=435.33;
  phi[2]=434.34;
  phi[3]=434.52;
  phi[4]=430.72;
  phi[5]=429.04;
  phi[6]=430.93;
  phi[7]=431.03;
  phi[8]=433.08;

  T = 0.4192;
  K = 435;

  opp=opprice(0,T,K,phi,8,dS,dt,dim,sigma_init);
  printf("\n%f\t%f",K,opp);

  free_vector(phi,1,8);
}
```

Program 2: Maximum likelihood method described in Section 4.2.

```
#include <math.h>
#include <stdio.h>
#include <stddef.h>
#include <stdlib.h>

#define ALF 1.0e-4
#define TOLX1 1.0e-7
#define ITMAX 10000
#define EPS 3.0e-8
#define TOLX2 (4*EPS)
#define STPMX 0.03

float FMAX(float a, float b) { if (a>b) return a; else return b;}
float FMIN(float a, float b) { if (a<b) return a; else return b;}

void lnsrch(int n, float xold[], float fold, float g[], float p[],
  float x[], float *f, float stpmax, int *check,
  float (*func)(float [], int, float [], float [], int, int),
  int M, float u[], int *roff, float modif[], int L, int Lmax) {

  int i;
  float a,alam,alam2,alamin,b,disc,f2,rhs1,rhs2,slope,sum,temp,
    test,tmplam;
  *check=0; *roff=0;
  for (sum=0.0,i=1;i<=n;i++) sum += p[i]*p[i];
  sum=sqrt(sum);
  if (sum > stpmax)
    for (i=1; i<=n; i++) p[i] *= stpmax/sum;
  for (slope=0.0, i=1; i<=n; i++) slope += g[i]*p[i];
  if (slope >=0.0) {
    for (i=1; i<=n; i++) x[i]=xold[i];
    *roff=1;
    *f=fold;
    return;
  }
  test=0.0;
  for (i=1; i<=n; i++) {
    temp=fabs(p[i])/FMAX(fabs(xold[i]),1.0);
    if (temp > test) test=temp;
  }
```

```
  alamin=TOLX1/test;
  alam=1.0;
  for (i=1; i<=n; i++)
    if (p[i] < 0) alam=FMIN(alam,(EPS-xold[i])/FMIN(p[i],-1.0));
  for (;;) {
    for (i=1; i<=n; i++) x[i]=xold[i]+alam*p[i];
    *f=(*func)(x,M,u,modif,L,Lmax);
    if (alam < alamin) {
      for (i=1; i<=n; i++) x[i]=xold[i];
      *check=1;
      return;
    } else if (*f <= fold+ALF*alam*slope) return;
    else {
      if (alam == 1.0) tmplam = -slope/(2.0*(*f-fold-slope));
      else {
        rhs1 = *f-fold-alam*slope;
        rhs2 = f2-fold-alam2*slope;
        a=(rhs1/(alam*alam)-rhs2/(alam2*alam2))/(alam-alam2);
        b=(-alam2*rhs1/(alam*alam)+alam*rhs2/(alam2*alam2))/(alam-alam2);
        if (a == 0.0) tmplam = -slope/(2.0*b);
        else {
          disc=b*b-3.0*a*slope;
          if (disc < 0.0) tmplam=0.5*alam;
          else if (b <= 0.0) tmplam=(-b+sqrt(disc))/(3.0*a);
          else tmplam=-slope/(b+sqrt(disc));
        }
        if (tmplam > 0.5*alam) tmplam=0.5*alam;
      }
    }
    alam2=alam;
    f2 = *f;
    alam=FMAX(tmplam,0.1*alam);
  }
}

void dpfmin(float p[], int n, float gtol, int *iter, float *fret,
  float (*func)(float [], int, float [], float [], int, int),
  void (*dfunc)(float [], float [], int, float [], float [], int, int),
  int M, float u[],int *roff, float modif[], int L, int Lmax) {

  void lnsrch(int n, float xold[], float fold, float g[], float p[],
    float x[], float *f, float stpmax, int *check,
    float (*func)(float [], int, float [], float [], int, int),
```

```
  int M, float u[], int *roff, float modif[], int L, int Lmax);
int check,i,its,j;
float den,fac,fad,fae,fp,stpmax,sum=0.0,sumdg,sumxi,temp,test;
float *dg,*g,*hdg,**hessin,*pnew,*xi;

dg=vector(1,n);
g=vector(1,n);
hdg=vector(1,n);
hessin=matrix(1,n,1,n);
pnew=vector(1,n);
xi=vector(1,n);
fp=(*func)(p,M,u,modif,L,Lmax);
(*dfunc)(p,g,M,u,modif,L,Lmax);
for (i=1; i<=n; i++) {
  for (j=1; j<=n; j++) hessin[i][j]=0.0;
  hessin[i][i]=1.0;
  xi[i] = -g[i];
  sum += p[i]*p[i];
}
stpmax=STPMX*FMAX(sqrt(sum),(float)n);
for (its=1; its<=ITMAX; its++) {
  *iter=its;
  lnsrch(n,p,fp,g,xi,pnew,fret,stpmax,&check,func,M,u,roff,
    modif,L,Lmax);
  fp = *fret;
  for (i=1; i<=n; i++) {
    xi[i]=pnew[i]-p[i];
    p[i]=pnew[i];
  }
  test=0.0;
  for (i=1; i<=n; i++) {
    temp=fabs(xi[i])/FMAX(fabs(p[i]),1.0);
    if (temp > test) test=temp;
  }
  if (test < TOLX2) {
    free_vector(xi,1,n);free_vector(pnew,1,n);
      free_matrix(hessin,1,n,1,n);
    free_vector(hdg,1,n);free_vector(g,1,n);free_vector(dg,1,n);
    return;
  }
  for (i=1; i<=n; i++) dg[i]=g[i];
  (*dfunc)(p,g,M,u,modif,L,Lmax);
  test=0.0;
```

```
  den=FMAX(*fret,1.0);
  for (i=1; i<=n; i++) {
    temp=fabs(g[i])*FMAX(fabs(p[i]),1.0)/den;
    if (temp > test) test=temp;
  }
  if (test < gtol) {
    free_vector(xi,1,n);free_vector(pnew,1,n);
    free_matrix(hessin,1,n,1,n);
    free_vector(hdg,1,n);free_vector(g,1,n);free_vector(dg,1,n);
    return;
  }
  for (i=1; i<=n; i++) dg[i]=g[i]-dg[i];
  for (i=1; i<=n; i++) {
    hdg[i]=0.0;
    for (j=1; j<=n; j++) hdg[i] += hessin[i][j]*dg[j];
  }
  fac=fae=sumdg=sumxi=0.0;
  for (i=1; i<=n; i++) {
    fac += dg[i]*xi[i];
    fae += dg[i]*hdg[i];
    sumdg += (dg[i])*(dg[i]);
    sumxi += (xi[i])*(xi[i]);
  }
  if (fac > sqrt(EPS*sumdg*sumxi)) {
    fac=1.0/fac;
    fad=1.0/fae;
    for (i=1; i<=n; i++) dg[i]=fac*xi[i]-fad*hdg[i];
    for (i=1; i<=n; i++) {
      for (j=i; j<=n; j++) {
        hessin[i][j] += fac*xi[i]*xi[j]-fad*hdg[i]*hdg[j]+
          fae*dg[i]*dg[j];
        hessin[j][i]=hessin[i][j];
      }
    }
  }
  for (i=1; i<=n; i++) {
    xi[i]=0.0;
    for (j=1; j<=n; j++) xi[i] -= hessin[i][j]*g[j];
  }
}
free_vector(xi,1,n);free_vector(pnew,1,n);free_matrix(hessin,1,n,1,n);
free_vector(hdg,1,n);free_vector(g,1,n);free_vector(dg,1,n);
return;
```

```
}

float func_old(float p[], int M, float u[], int L, int Lmax) {
  float sigma,sum=0.0,v=0.0;
  int i,k;
  sigma = u[L]*u[L];
  if (L == Lmax) sum = log(sigma)+u[L]*u[L]/sigma;
  for (i=L+1; i<=M; i++) {
    if (i == L+1)
      for (k=1; k<=L; k++) v +=u[k];
    else v += u[i-1]-u[i-L-1];
    sigma = p[1]+p[2]*(v*v/L)+p[3]*sigma;
    if (sigma <= 0) printf("variance is negative in func");
    if (i >= Lmax) sum += log(sigma)+u[i]*u[i]/sigma;
  }
  return sum;
}

void dfunc_old(float p[], float g[], int M, float u[], int L, int Lmax) {
  float sigma,term,A,B,C,v=0.0;
  int i,k;

  g[1]=g[2]=g[3]=0.0;
  sigma=u[L]*u[L];
  A=1;
  B=C=u[L]*u[L];
  for (i=L+1; i<=M; i++) {
    if (i == L+1)
      for (k=1; k<=L; k++) v +=u[k];
    else v += u[i-1]-u[i-L-1];
    sigma = p[1]+p[2]*(v*v/L)+p[3]*sigma;
    if (sigma <= 0) printf("variance is negative in dfunc");
    if (i >= Lmax) {
      term = (1/sigma)-(u[i]*u[i])/(sigma*sigma);
      g[1] += term*A;
      g[2] += term*B;
      g[3] += term*C;
    }
    A = 1+p[3]*A;
    B = v*v+p[3]*B;
    C = sigma+p[3]*C;
  }
}
```

```
void stock_in(char filename[], int M, float S[]) {
  FILE *stream;
  float a;
  int i;
  if ((stream=fopen(filename,"r")) == NULL) printf("Cannot open file");
    for (i=1; i<=M; i++) {
    fscanf(stream,"%f",&a);
    S[i] = a;
  }
  fclose(stream);
}

float func(float p[], int M, float u[], float modif[], int L, int Lmax) {
  int i; float temp;
  for (i=1; i<=3; i++) p[i]=p[i]/modif[i];
  temp=func_old(p,M,u,L,Lmax);
  for (i=1; i<=3; i++) p[i]=p[i]*modif[i];
  return temp;
}

void dfunc(float p[], float g[], int M, float u[], float modif[],
  int L, int Lmax) {

  int i; for (i=1; i<=3; i++) p[i]=p[i]/modif[i];
  dfunc_old(p,g,M,u,L,Lmax);
  for (i=1; i<=3; i++) p[i]=p[i]*modif[i];
}

void dpfmin_mod(float p[], int n, float gtol, int *iter, float *fret,
  float (*func)(float [], int, float [], float [], int, int),
  void (*dfunc)(float [], float [], int, float [], float[], int, int),
  int M, float u[],int *roff, float modif[], int L, int Lmax) {

  int i; for (i=1; i<=n; i++) p[i]=p[i]*modif[i];
  dpfmin(p,n,gtol,iter,fret,func,dfunc,M,u,roff,modif,L,Lmax);
  for (i=1; i<=n; i++) p[i]=p[i]/modif[i];
}

int main(int argc, char *argv[]) {
  float *p,fret,*st,*u,p1,p2,p3,pp1,pp2,pp3,ftemp,*modif,
    ftemp2=0.0,q1,q2,q3;
  int iter,M,i,roff,proff,nroff,ntot,totiter,L,Lmax,Lout;
```

```
FILE *fout;

printf("Data: >%s\n",argv[1]);

M = atoi(argv[2]);
Lmax = atoi(argv[3]);

st=vector(1,M);
u=vector(1,M-1);
p=vector(1,3);
modif=vector(1,3);

if ((fout=fopen(argv[4],"w")) == NULL) printf("Cannot open file");

stock_in(argv[1],M,st);
for (i=1; i<=M-1; i++) u[i] = (st[i+1]/st[i])-1;
for (L=1; L<=Lmax; L++) {
  proff=0; nroff=0; ntot=0; totiter=0; ftemp=0.0;
  modif[1]=10000; modif[2]=modif[3]=1;

  for (p1=0.000003; p1<=0.000100; p1 += 0.000010) {
    for (p2=0.02; p2<=0.95; p2 += 0.1) {
      for (p3=0.02; p2+p3<=0.99; p3 += 0.1) {
        p[1]=p1;
        p[2]=p2;
        p[3]=p3;
        dpfmin_mod(p,3,0.0001,&iter,&fret,func,dfunc,M-1,u,
          &roff,modif,L,Lmax);
        nroff += roff;
        ntot++;
        totiter += iter;
        if ((ftemp < -fret) & (p[2]+p[3] < 0.99)) {
          ftemp = -fret;
          pp1 = p[1];
          pp2 = p[2];
          pp3 = p[3];
          proff = roff;
        }
      }
    }
  }

  fprintf(fout,"%d\t%1.4f\t%1.4f\t%1.4f\t%5.2f\n",L,
```

```
      sqrt(pp1/(1-pp2-pp3)),pp2,pp3,ftemp);
    if (ftemp2 < ftemp) {
      ftemp2 = ftemp;
      q1 = pp1;
      q2 = pp2;
      q3 = pp3;
      Lout = L;
    }
  }

  printf("\nThe maximal value of MLE is %f\n\tL=%d, sigma=%1.4f,
    alpha=%1.4f,beta=%1.4f\n\n",ftemp2,Lout,sqrt(252*q1/(1-q2-q3)),
    q2,q3);
  free_vector(p,1,3);
  free_vector(modif,1,3);
  free_vector(st,1,M);
  free_vector(u,1,M-1);
  fclose(fout);
}
```

Strike price	375	415	435	450	450
Implied volat.	0.1238	0.1124	0.1106	0.1110	0.1142

Table 1: Monte Carlo simulation results for continuous-time GARCH (2.14) with $\alpha = 14.375$ and $\gamma = 13.475$.

Strike price	Simulation	FDM
375	0.1040	0.1021
415	0.1036	0.1044
435	0.1035	0.1041
450	0.1035	0.1040
475	0.1036	0.1032

Table 2: Implied volatility for stochastic volatility model (3.12) with $\alpha = 0.0575$ and $\gamma = 0.0539$: a comparison of Monte Carlo simulation results with the finite difference method (FDM) for general equation.

Strike price	Simulation	FDM
375	0.1277	0.1207
415	0.1122	0.1068
435	0.1104	0.1090
450	0.1113	0.1039
475	0.1163	0.0839

Table 3: Implied volatility for stochastic volatility model (3.12) with $\alpha = 14.375$ and $\gamma = 13.475$: a comparison of Monte Carlo simulation results with the finite difference method (FDM) for general equation.

Year	l	$\sqrt{V}$	α	β	γ
1990	1	0.1873	0.0620	0.8443	0.0937
1991	15	0.1603	0.5663	0.1131	0.3206
1992	–	–	–	–	–
1993	1	0.0714	0.0403	0.8073	0.1524
1992-93	4	0.0857	0.0446	0.8505	0.1049
1990-93	7	0.1186	0.0575	0.8886	0.0539

Table 4: Results of ML-AICC method of parameter estimation applied to S&P500 data.

Lag k	η_k=autocorr. $\{\varepsilon_n\}$	ϕ_k=autocorr. $\{\varepsilon_n^2\}$	θ_k=autocorr. $\{\varepsilon_n^2/\sigma_n^2\}$
1	0.0310	0.0429	-0.0346
2	-0.0454	0.1325	0.0188
3	0.0084	0.0762	0.0553
4	-0.0053	0.1225	0.0045
5	0.0188	0.0779	0.0231
6	-0.0305	0.0971	-0.0001
7	-0.0957	0.0604	-0.0279
8	-0.0021	0.0369	0.0038
9	0.0494	0.0961	0.0148
10	-0.0242	0.1009	0.0301
11	0.0280	0.0566	-0.0254
12	0.0439	0.0074	-0.0336
13	0.0360	0.2219	0.0708
14	0.0204	0.0746	-0.0054
15	-0.0087	0.1402	0.0071

Table 5: Autocorrelation structure in the dataset for 1990-1993.

r	μ	OP	OP Error
0.01	0.01	12.310362	0.018819
0.01	0.02	12.308543	0.018845
0.01	0.03	12.308754	0.018881
0.01	0.04	12.310895	0.018926
0.01	0.05	12.315323	0.018980
0.02	0.01	13.210551	0.019086
0.02	0.02	13.208350	0.019110
0.02	0.03	13.208447	0.019141
0.02	0.04	13.210579	0.019181
0.02	0.05	13.214739	0.019231
0.03	0.01	14.147881	0.019064
0.03	0.02	14.145847	0.019086
0.03	0.03	14.147392	0.019361
0.03	0.04	14.149373	0.019396
0.03	0.05	14.153369	0.019441
0.04	0.01	15.125459	0.019250
0.04	0.02	15.123305	0.019268
0.04	0.03	15.123144	0.019295
0.04	0.04	15.124992	0.019330
0.04	0.05	15.130386	0.019616
0.05	0.01	16.140877	0.019400
0.05	0.02	16.138606	0.019414
0.05	0.03	16.138304	0.019436
0.05	0.04	16.139971	0.019467
0.05	0.05	16.143603	0.019508

Table 6: European call option price (OP) for different values of r and μ. All the other parameters are fixed.

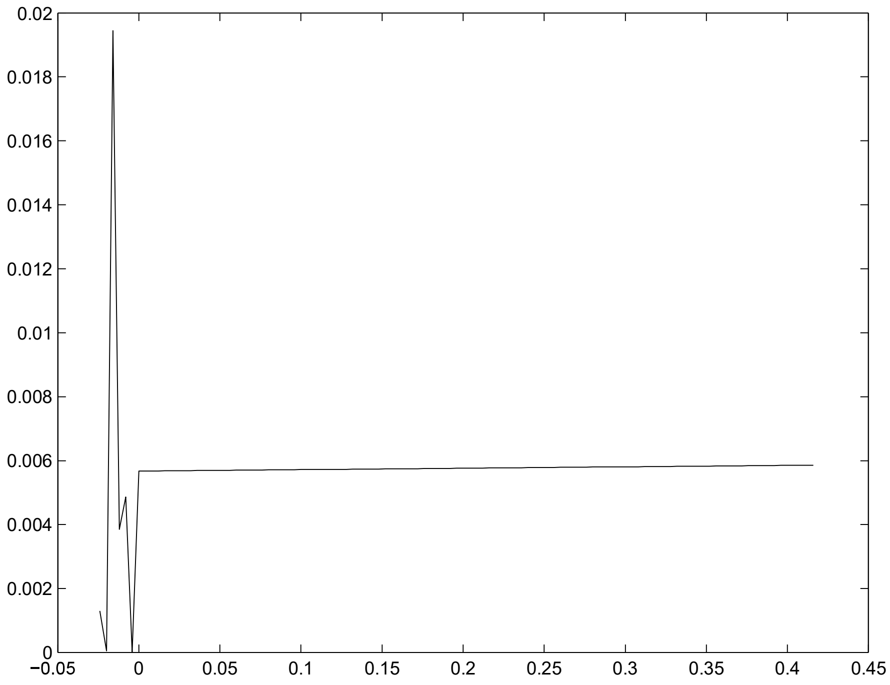

Figure 1: Solution of FDE (2.16) vs. time.

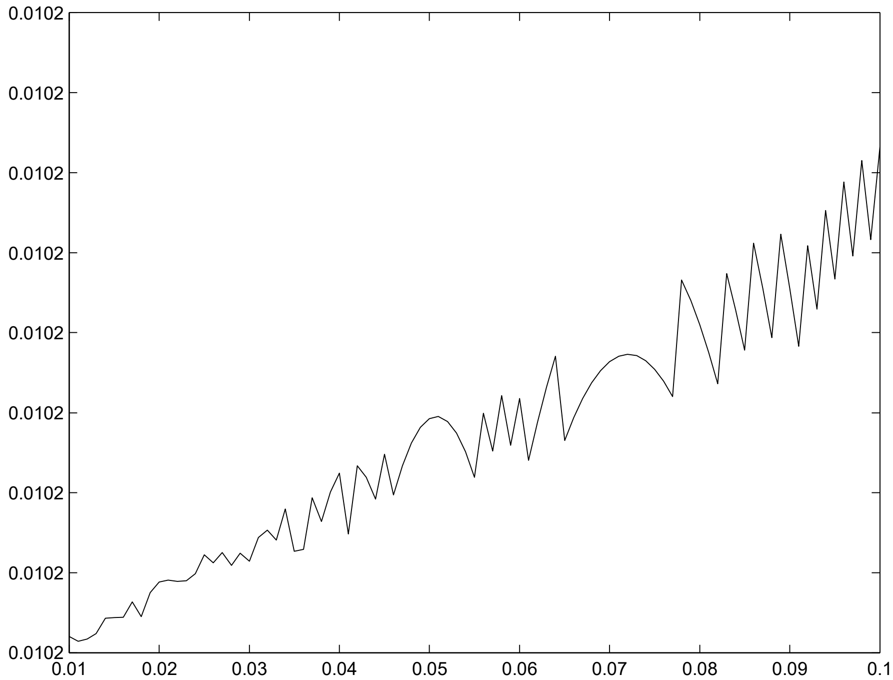

Figure 2: Dependence of variance $v(T)$ on delay τ.

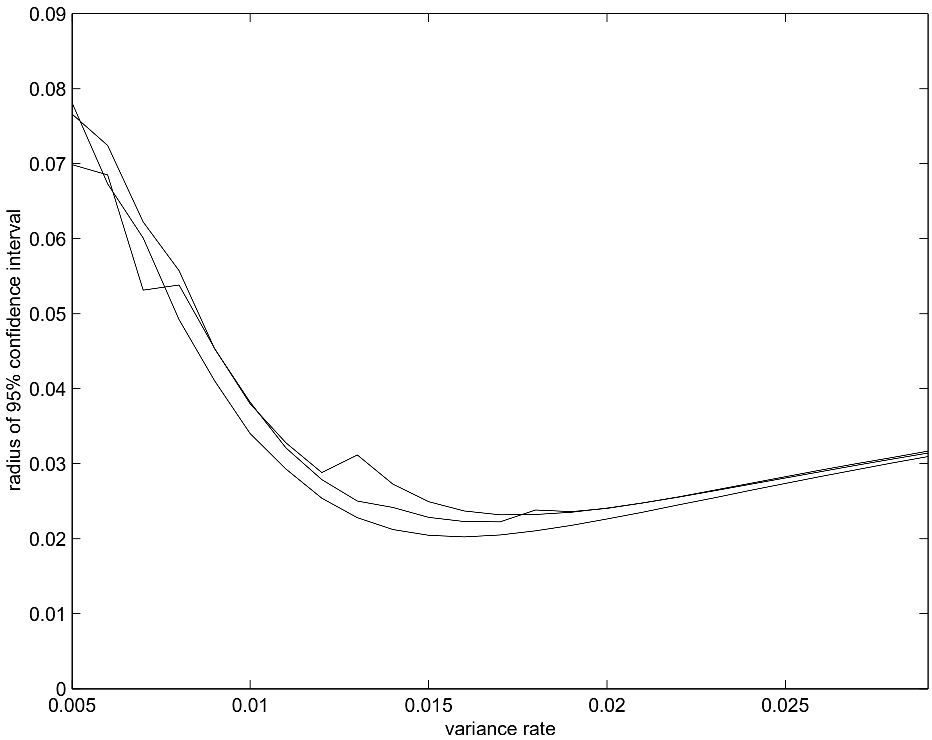

Figure 3: Radius of 95% confidence interval of the Monte Carlo estimator vs. variance rate V_{BS} used in the option price approximation. Several realizations are shown.

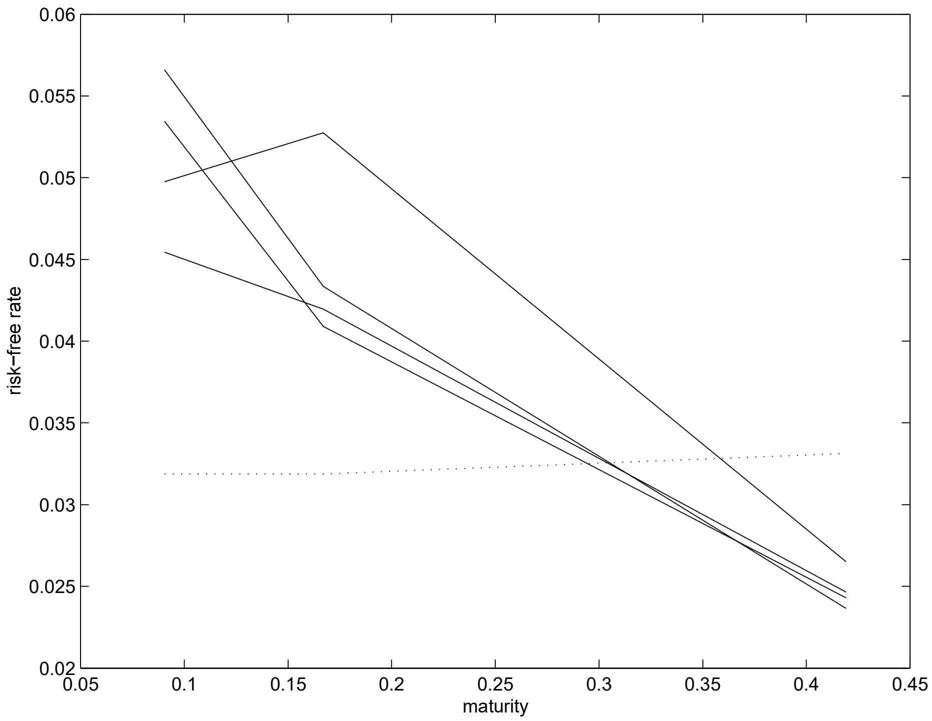

Figure 4: Constant risk-free rate r implied from the market prices vs. maturity for different strike prices. The estimation is based on GARCH stochastic volatility model for S&P500 index.

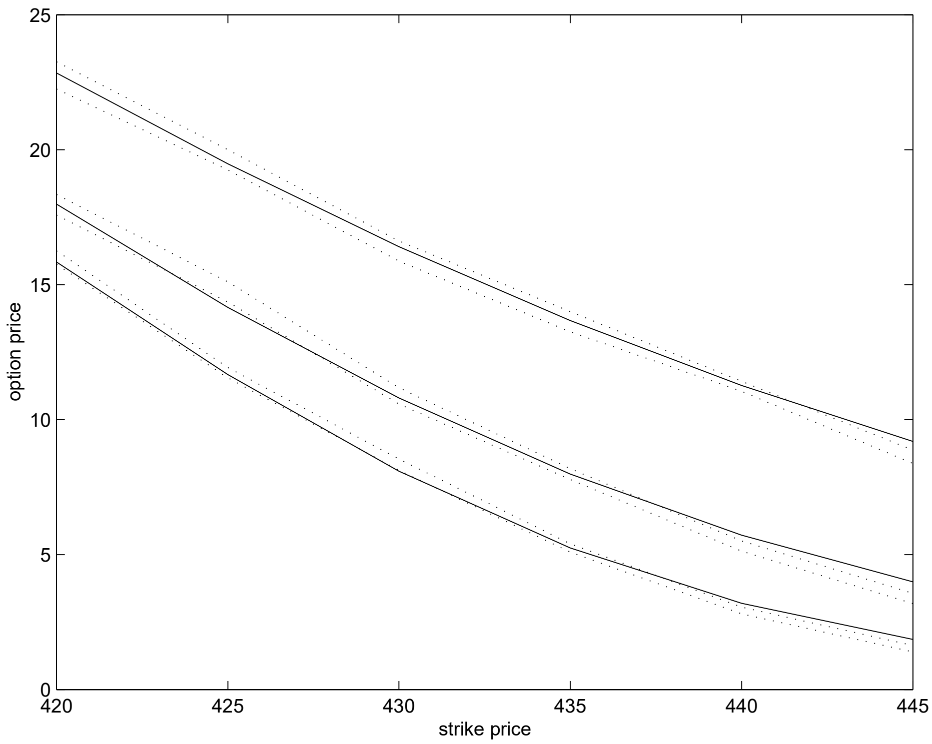

Figure 5: European call option price vs. strike price for three different maturities. Dashed lines represent bid and ask prices observed in the market and solid lines represent simulated prices based on CIR-GARCH stochastic volatility model.

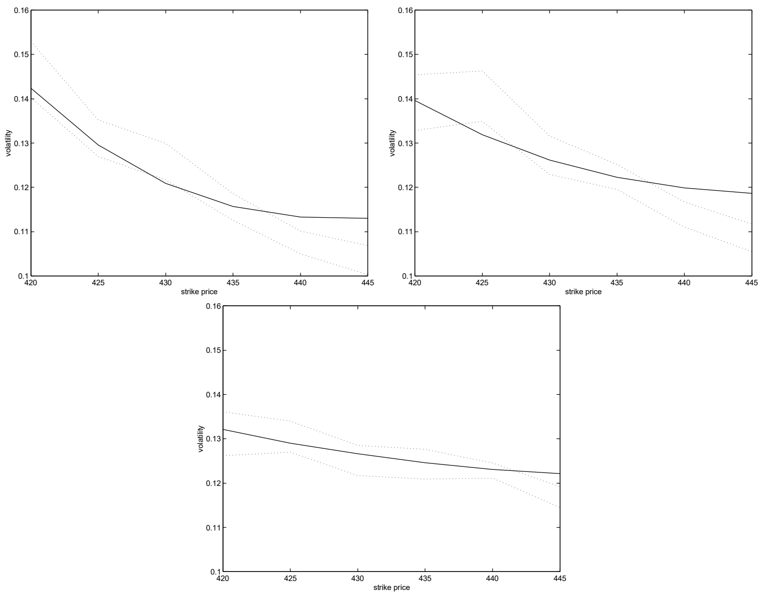

Figure 6: Implied volatility vs. strike price for three different maturities. The volatility plots correspond to the option prices depicted on Figure 5.

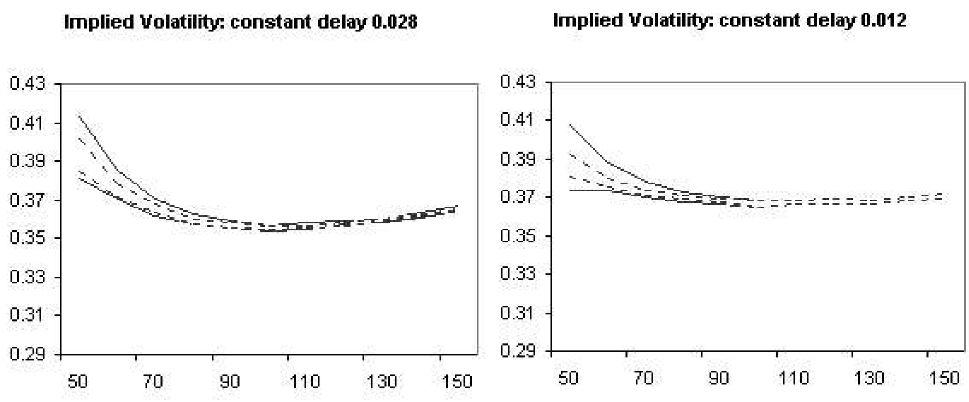

Figure 7: Implied volatility for models with constant delay.

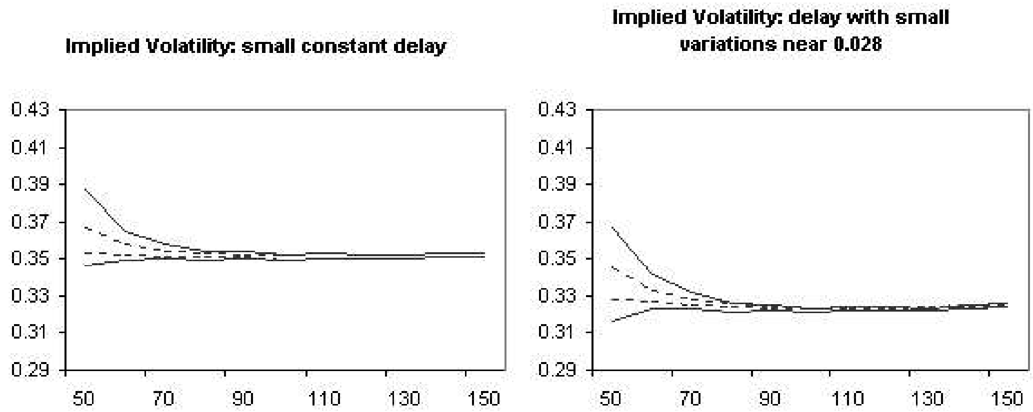

Figure 8: Implied volatility for models with nearly constant delay.

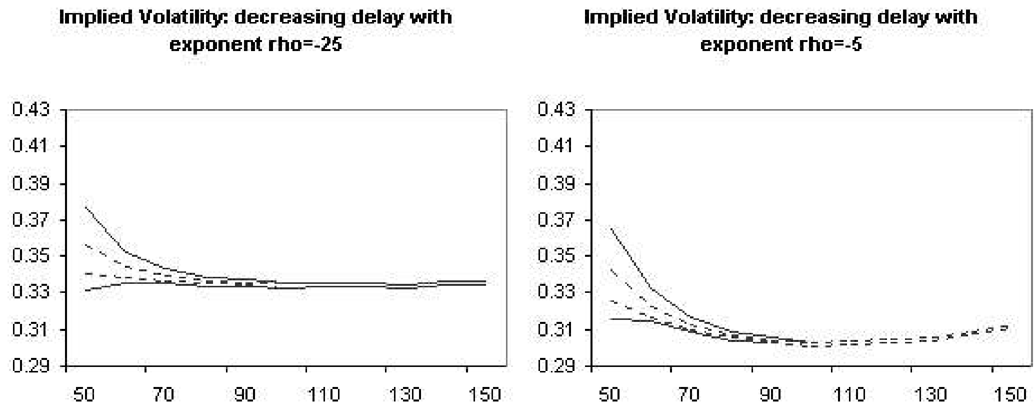

Figure 9: Implied volatility for models with decreasing state-dependent delay.

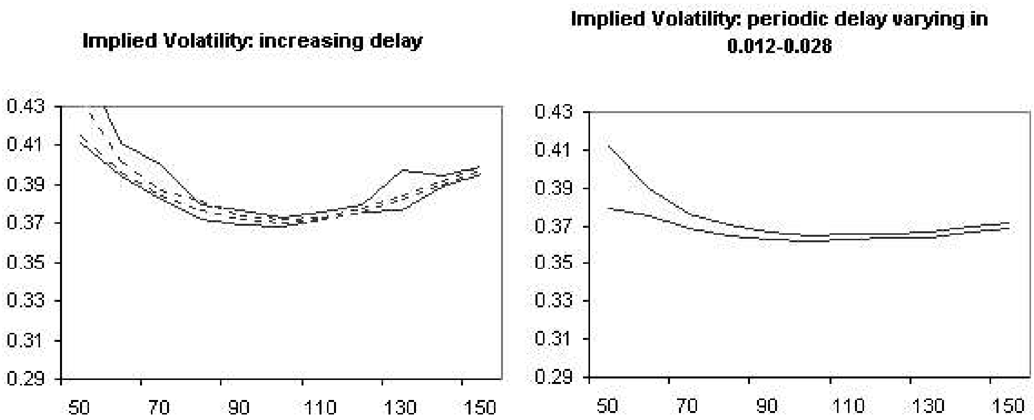

Figure 10: Implied volatility for models with various state-dependent delays.