Stochastic State-dependent Delay Differential Equations with Applications in Finance *

Y. I. Kazmerchuk †and J. H. Wu ‡

November 2003

Abstract

In this paper, we considered a new class of stochastic differential equations involving delayed response, where the delay depends on the system's state. We obtained results for the existence and uniqueness of solutions, and we proved that the Euler discrete-time approximation scheme is convergent with a strong order of convergence. We used the approximation result to simulate the continuous-time GARCH(1,1) model for stochastic volatility with state-dependent delay. The simulation results showed that a choice of state-dependent delay function spans a wide variety of U-shaped implied volatility plots, and the state-dependence can also be used to control the height of the plots.

Key words: state-dependent delay, Euler discrete-time approximation, implied volatility, continuous-time GARCH model.

AMS(MOS) Subject Classifications: 34K50, 62P05

*Supported by NSERC, CRC and MITACS

†yorik@mathstat.yorku.ca

‡wujh@mathstat.yorku.ca

Department of Mathematics & Statistics, York University, Toronto, Canada M3J 1P3

1 Introduction

In [1], we considered the following stochastic volatility model

$$
\begin{aligned}
dx(t) &= rx(t)\ dt + \sqrt{y(t)}x(t)\ dW(t),\\
\frac{dy(t)}{dt} &= \gamma V + \frac{\alpha}{\tau}\left\{\ln\frac{x(t)}{x(t-\tau)} - \mu\tau + \frac{1}{2}\int_{t-\tau}^{t} y(s)\ ds\right\}^2 - (\alpha+\gamma)y(t),
\end{aligned}
$$

as a continuous-time limit of the well-known GARCH(1,1) model. Here $x(t)$ represents the stock price and $\sqrt{y(t)}$ represents its volatility, that is the standard deviation of $\log x(t)$. The time delay parameter τ was considered as a constant. An equation for European call option was derived and a numerical scheme was introduced. Some simulation and numerical results showed that the model produces a U-shaped implied volatility smile. On the other hand, via the employed parameter estimation, we showed that the delay varies considerably from year to year. This suggests that the delay involved in the market response may not be a constant but may depend on the stock price.

In this paper, we consider the case where the time delay depends on x and y. In particular, we assume that τ is a decreasing function of y, due to the following empirical observation: since y represents volatility of the stock, the greater y the more volatile price, and therefore, the more active trading. On the other hand, it is reasonable to assume that the market's response to changes in the stock price is faster when the volatility is higher. As a consequence, the delay is a decreasing function of the volatility.

This leads us to consider a general stochastic state-dependent delay differential equation (SSDDE)

$$
\begin{cases}
dX(t) = F(X(t), X(t-\tau))\ dt + G(X(t), X(t-\tau))\ dW(t),\\
X(t) = \varphi(t), \quad t\in[-\delta, 0],
\end{cases}
$$

where τ takes values in $[\delta_0, \delta]$ and $\tau = \tau(X(t-\kappa))$ for $\kappa \in [\delta_0, \delta]$.

We shall establish a result on the existence of a solution to the above SSDDE. We shall also prove that the Euler discrete-time approximation scheme has a strong order of convergence, and this convergence also yields the uniqueness of a solution. We shall apply the approximation result to our continuous-time GARCH model with state-dependent delay, and we shall use the Monte-Carlo method to carry out the simulation and to analyze the implied volatility structure.

2 Existence

Here, we shall establish the existence of a solution to the following multi-dimensional SSDDE:

$$
\begin{cases} dX(t) = F(X(t), X(t-\tau))\, dt + G(X(t), X(t-\tau))\, dW(t), \\ X(t) = \varphi(t), \quad t \in [-\delta, 0], \end{cases} \tag{1}
$$

where $\tau = \tau(X(t), X(t-\kappa))$, $0 < \delta_0 \leq \tau(s_1, s_2) \leq \delta$ for $s_1, s_2 \in \mathbf{R}^n$, $\kappa \in [\delta_0, \delta]$ and $\{W(t)\}$ is a Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

In what follows, $|\cdot|$ is the Euclidean norm. For any $a < b$, $L_2(\Omega, C[a,b])$ is the space of $C[a,b]$-valued random variables equipped with the norm defined by

$$\|\eta\|_L := \sqrt{E\|\eta\|_c^2}\,,$$

where $\|\eta\|_c = \sup_{a \leq t \leq b} |\eta(t)|$.

For $\alpha \in (0,1]$, we define a set $S_\alpha \subset L_2(\Omega, C[-\delta, 0])$ by

$$S_\alpha := \left\{\eta \mid \exists M\ \forall t_1, t_2 \in [-\delta, 0]:\ E|\eta(t_1) - \eta(t_2)|^2 \leq M|t_1 - t_2|^{2\alpha}\right\}.$$

Theorem 1 *Assume F, G and τ are continuous in their arguments. Then for any $\mathcal{F}_0$-measurable initial data $\varphi \in L_2(\Omega, C[-\delta, 0])$ there exists a solution of SSDDE (1) defined on $[0, +\infty)$.*

Proof: We use the so-called method of steps to construct a solution to (1). Note that for any $t \in [n\delta_0, (n+1)\delta_0]$, $n \geq 0$ we have the following.

$$
\begin{aligned} X(t) =& X(n\delta_0) + \int_{n\delta_0}^t F(X(s), X(s - \tau(X(s), X(s-\kappa))))\, ds \\ &+ \int_{n\delta_0}^t G(X(s), X(s - \tau(X(s), X(s-\kappa))))\, dW(s). \end{aligned} \tag{2}
$$

Since $\tau(x,y) \geq \delta_0$ for $x, y \in \mathbf{R}^n$, we have $s - \tau(X(s), X(s-\kappa)) \leq s - \delta_0$ and (2) becomes a stochastic ODE. Note that $\{X(u), -\delta \leq u \leq s - \delta_0\}$ is a.s. continuous, and therefore there is an a.s. continuous solution to (2) defined on $[n\delta_0, (n+1)\delta_0]$ (see [4]).

3 Discrete-time Approximations of SSDDEs

In this section we prove that the Euler discrete-time scheme for a SSDDE with a special form has 1/2-strong order of convergence over $[0, T]$. This extends the result for stochastic ODEs. We also show that the scheme for a slightly more restrictive SSDDE has $2^{-[\frac{T}{\kappa}+2]}$-strong order of convergence, and we shall derive the uniqueness of solutions as a corollary.

3.1 SSDDE: Type I

We consider the following special case of (1).

$$
(3)\quad \begin{cases} dX_1(t) = f(X(t), X_2(t-\tau))\, dt + g(X(t), X_2(t-\tau))\, dW(t), \\ dX_2(t) = z(X(t), X_2(t-\tau))\, dt, \\ X(t) = [X_1(t), X_2(t)]^T = \varphi(t), \quad t \in [-\delta, 0], \end{cases}
$$

where $\tau = \tau(X(t))$, $X_1(t) \in \mathbf{R}^{n_1}$, $X_2(t) \in \mathbf{R}^{n_2}$ and $X(t) \in \mathbf{R}^{n_1+n_2}$. Note that only the X_2-component has state-dependent delayed effect.

For a fixed $h > 0$ and $t \in \mathbf{R}$, we denote $\lfloor t \rfloor = h[t/h]$, where $[\cdot]$ is the integer part. *Strong Euler approximation scheme* for (3) is defined as follows:

(4)

$$
\begin{cases} d\bar{X}_1(t) = f(\bar{X}(\lfloor t \rfloor), \bar{X}_2(\lfloor t \rfloor - \lfloor \bar{\tau} \rfloor))dt + g(\bar{X}(\lfloor t \rfloor), \bar{X}_2(\lfloor t \rfloor - \lfloor \bar{\tau} \rfloor))dW(t), \\ d\bar{X}_2(t) = z(\bar{X}(\lfloor t \rfloor), \bar{X}_2(\lfloor t \rfloor - \lfloor \bar{\tau} \rfloor))dt, \\ \bar{X}(t) = [\bar{X}_1(t), \bar{X}_2(t)]^T = \varphi(t), \quad t \in [-\delta, 0], \end{cases}
$$

where $\lfloor \bar{\tau} \rfloor = \lfloor \tau(\bar{X}(\lfloor t \rfloor)) \rfloor$.

Theorem 2 *Assume that f, g, z and τ are Lipschitz continuous with respect to all of their arguments. Then for any $\mathcal{F}_0$-measurable initial data $\varphi \in S_1$, there exists a constant $C(T, \varphi)$ such that*

$$
\sup_{t\in[0,T]} E|X(t) - \bar{X}(t)|^2 \le C(T, \varphi)\, h
$$

for sufficiently small h, where h is the partition's mesh size, X and $\bar{X}$ satisfy (3) and (4) respectively. Moreover, the solution X of (3) is pathwise unique.

Proof: Using representations (3) and (4) for X and $\bar{X}$, we get

(5)

$$
\begin{aligned}
\|X - \bar{X}\|^2_{L_2(\Omega, C[0,t])} &= \sup_{u\in[0,t]} E|X(u) - \bar{X}(u)|^2 \le \\
&\le 2 \sup_{u\in[0,t]} E\left|\int_0^u (f(X(s), X_2(s-\tau)) - f(\bar{X}(\lfloor s \rfloor), \bar{X}_2(\lfloor s \rfloor - \lfloor \bar{\tau} \rfloor)))ds\right|^2 \\
&+ 2 \sup_{u\in[0,t]} E\left|\int_0^u (g(X(s), X_2(s-\tau)) - g(\bar{X}(\lfloor s \rfloor), \bar{X}_2(\lfloor s \rfloor - \lfloor \bar{\tau} \rfloor)))dW(s)\right|^2 \\
&+ \sup_{u\in[0,t]} E\left|\int_0^u (z(X(s), X_2(s-\tau)) - z(\bar{X}(\lfloor s \rfloor), \bar{X}_2(\lfloor s \rfloor - \lfloor \bar{\tau} \rfloor)))ds\right|^2
\end{aligned}
$$

$$\leq 2t \int_0^t E|f(X(s), X_2(s-\tau)) - f(\bar{X}(\lfloor s \rfloor), \bar{X}_2(\lfloor s \rfloor - \lfloor \bar{\tau} \rfloor))|^2 ds$$
$$+ 2\int_0^t E|g(X(s), X_2(s-\tau)) - g(\bar{X}(\lfloor s \rfloor), \bar{X}_2(\lfloor s \rfloor - \lfloor \bar{\tau} \rfloor))|^2 ds$$
$$+ t\int_0^t E|z(X(s), X_2(s-\tau)) - z(\bar{X}(\lfloor s \rfloor), \bar{X}_2(\lfloor s \rfloor - \lfloor \bar{\tau} \rfloor))|^2 ds.$$

We now estimate each term in (5). First of all, we have

$$\int_0^t E|f(X(s), X_2(s-\tau)) - f(\bar{X}(\lfloor s \rfloor), \bar{X}_2(\lfloor s \rfloor - \lfloor \bar{\tau} \rfloor))|^2 ds \leq 5\left[J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t)\right],$$

where

$$J_1(t) = \int_0^t E|f(X(s), X_2(s-\tau)) - f(X(\lfloor s \rfloor), X_2(s-\tau))|^2\, ds,$$
$$J_2(t) = \int_0^t E|f(X(\lfloor s \rfloor), X_2(s-\tau)) - f(\bar{X}(\lfloor s \rfloor), X_2(s-\tau))|^2\, ds,$$
$$J_3(t) = \int_0^t E|f(\bar{X}(\lfloor s \rfloor), X_2(s-\tau)) - f(\bar{X}(\lfloor s \rfloor), X_2(\lfloor s \rfloor - \lfloor \tau \rfloor))|^2\, ds,$$
$$J_4(t) = \int_0^t E|f(\bar{X}(\lfloor s \rfloor), X_2(\lfloor s \rfloor - \lfloor \tau \rfloor)) - f(\bar{X}(\lfloor s \rfloor), X_2(\lfloor s \rfloor - \lfloor \bar{\tau} \rfloor))|^2\, ds,$$
$$J_5(t) = \int_0^t E|f(\bar{X}(\lfloor s \rfloor), X_2(\lfloor s \rfloor - \lfloor \bar{\tau} \rfloor)) - f(\bar{X}(\lfloor s \rfloor), \bar{X}_2(\lfloor s \rfloor - \lfloor \bar{\tau} \rfloor))|^2\, ds.$$

Here, $\tau = \tau(X(s))$, $\lfloor \tau \rfloor = \lfloor \tau(X(\lfloor s \rfloor)) \rfloor$ and $\lfloor \bar{\tau} \rfloor = \lfloor \tau(\bar{X}(\lfloor s \rfloor)) \rfloor$. Since f and τ are Lipschitz continuous and X is $1/2$-Lipschitz continuous with the constant $M(\varphi)$ (as in definition of $S_{1/2}$), we obtain

$$J_1(t) \leq L\int_0^t E|X(s) - X(\lfloor s \rfloor)|^2\, ds \leq L\int_0^t M(\varphi)(s - \lfloor s \rfloor)\, ds$$
$$\leq LtM(\varphi)\, h \equiv C_1(t,\varphi)\, h,$$
$$J_2(t) \leq L\int_0^t E|X(\lfloor s \rfloor) - \bar{X}(\lfloor s \rfloor)|^2\, ds \leq L\int_0^t \|X - \bar{X}\|^2_{L_2(\Omega, C[0,s])}\, ds,$$
$$J_3(t) \leq L\int_0^t E|X_2(s-\tau) - X_2(\lfloor s \rfloor - \lfloor \tau \rfloor)|^2\, ds$$
$$\leq LM_2(\varphi)\int_0^t E|s - \tau - \lfloor s \rfloor + \lfloor \tau \rfloor|^2\, ds$$

$$\leq 2LM_2(\varphi)\int_0^t (s-\lfloor s\rfloor)^2\,ds + 2LM_2(\varphi)\int_0^t E(\tau-\lfloor\tau\rfloor)^2\,ds$$
$$\leq 6LM_2(\varphi)t\,h^2 + 4LM_2(\varphi)L_\tau\int_0^t E|X(s)-X(\lfloor s\rfloor)|^2\,ds$$
$$\leq 6LM_2(\varphi)t\,h^2 + 4LM_2(\varphi)L_\tau M(\varphi)t\,h \equiv C_2(t,\varphi)\,h^2 + C_3(t,\varphi)\,h.$$

Here, $\sqrt{L}$ and $\sqrt{L_\tau}$ are the Lipschitz constants of f and τ, respectively. We also used the fact that φ and X_2 are Lipschitz continuous with constant $M_2(\varphi)$ because the diffusion coefficient of second equation in (3) is zero. We also have

$$J_4(t) \leq LM_2(\varphi)\int_0^t E|\lfloor\tau\rfloor - \lfloor\bar\tau\rfloor|^2\,ds$$
$$\leq 2LM_2(\varphi)t\,h^2 + 2LM_2(\varphi)L_\tau\int_0^t E|X(\lfloor s\rfloor) - \bar X(\lfloor s\rfloor)|^2\,ds$$
$$\leq 2LM_2(\varphi)t\,h^2 + 2LM_2(\varphi)L_\tau\int_0^t \|X-\bar X\|^2_{L_2(\Omega,C[0,s])}\,ds$$
$$\equiv C_4(t,\varphi)\,h^2 + C_5(\varphi)\int_0^t \|X-\bar X\|^2_{L_2(\Omega,C[0,s])}\,ds,$$
$$J_5(t) \leq L\int_0^t E|X_2(\lfloor s\rfloor - \lfloor\bar\tau\rfloor) - \bar X_2(\lfloor s\rfloor - \lfloor\bar\tau\rfloor)|^2\,ds$$
$$\leq L\int_0^t \|X-\bar X\|^2_{L_2(\Omega,C[0,s])}\,ds.$$

Therefore,

$$\int_0^t E|f(X(s),X_2(s-\tau)) - f(\bar X(\lfloor s\rfloor),\bar X_2(\lfloor s\rfloor - \lfloor\bar\tau\rfloor))|^2 ds$$
$$\leq C_6(t,\varphi)\,h + C_7(\varphi)\int_0^t \|X-\bar X\|^2_{L_2(\Omega,C[0,s])}\,ds$$

for sufficiently small h. Carrying out the same analysis for terms in (5) with g and z, we obtain

$$\|X-\bar X\|^2_{L_2(\Omega,C[0,t])} \leq A(\varphi,T)\int_0^t \|X-\bar X\|^2_{L_2(\Omega,C[0,s])}\,ds + B(T,\varphi)\,h \tag{6}$$

for certain constants A and B. Consequently, an application of the Grownwall inequality yields

$$\|X-\bar X\|^2_{L_2(\Omega,C[0,T])} \leq B(T,\varphi)e^{A(\varphi,T)T}\,h.$$

Uniqueness of the solution follows from this inequality, and the theorem is proved.

3.2 SSDDE: Type II

In this subsection, we consider SSDDE of the following more general form:

$$(7)\quad \begin{cases} dX(t) = F(X(t), X(t-\tau))\, dt + G(X(t), X(t-\tau))\, dW(t), \\ X(t) = \varphi(t), \quad t \in [-\delta, 0], \end{cases}$$

where $\tau = \tau(X(t-\kappa))$. Note that τ depends on κ-delayed value of X only. The Euler discrete-time scheme for (7) is given by

(8)

$$\begin{cases} d\bar{X}(t) = F(\bar{X}(\lfloor t \rfloor), \bar{X}(\lfloor t \rfloor - \lfloor \bar{\tau} \rfloor))\, dt + G(\bar{X}(\lfloor t \rfloor), \bar{X}(\lfloor t \rfloor - \lfloor \bar{\tau} \rfloor))\, dW(t), \\ \bar{X}(t) = \varphi(t), \quad t \in [-\delta, 0], \end{cases}$$

where $\lfloor \bar{\tau} \rfloor = \lfloor \tau(\bar{X}(\lfloor t - \kappa \rfloor)) \rfloor$.

Theorem 3 *Assume F, G and τ are Lipschitz continuous with respect to all of their arguments. Then for any $\mathcal{F}_0$-measurable initial data $\varphi \in S_{1/2}$, there exists a constant $C(T, \varphi)$ such that*

$$\sup_{t \in [0,T]} E|X(t) - \bar{X}(t)|^2 \leq C(T, \varphi)\; h^n$$

for sufficiently small h, where h is the partition's mesh size, $n = 2^{-[\frac{T}{\kappa}]-1}$. Moreover, the solution X of (7) is pathwise unique.

Proof: We use similar arguments used in the proof of Theorem 2, with necessary modifications. Since X is 1/2-Lipschitz continuous, estimation for $J_3(t)$ and $J_4(t)$ becomes

$$\begin{aligned} J_4(t) &\leq LM(\varphi) \int_0^t E|\lfloor \tau \rfloor - \lfloor \bar{\tau} \rfloor|\, ds \\ &\leq LM(\varphi) t\; h + LM(\varphi) L_\tau \int_0^t E|X(\lfloor s - \kappa \rfloor) - \bar{X}(\lfloor s - \kappa \rfloor)|\, ds \\ &\leq LM(\varphi) t\; h + LM(\varphi) L_\tau \int_0^t \|X - \bar{X}\|_{L_2(\Omega, C[-\delta, s-\kappa])}\, ds \\ &\equiv C_4(t, \varphi)\; h + C_5(\varphi) \int_0^t \|X - \bar{X}\|_{L_2(\Omega, C[-\delta, s-\kappa])}\, ds, \end{aligned}$$

and

$$J_3(t) \leq C_2(t, \varphi) h + C_3(t, \varphi)\sqrt{h}.$$

In addition, we should replace (6) by

$$\varepsilon(t) \leq A(\varphi, T) \int_0^t \left(\varepsilon(s) + \sqrt{\varepsilon(s-\kappa)} \right)\, ds + B(T, \varphi)\, \sqrt{h},$$

where $\varepsilon(t) = \|X - \bar{X}\|^2_{L_2(\Omega, C[-\delta,t])}$. Since X and $\bar{X}$ have the same initial data, $\varepsilon(t) = 0$ for $t \le 0$. By Grownwall inequality, we then obtain

$$\varepsilon(t) \le Be^{A\kappa}\sqrt{h} \quad \text{for } t \in [0, \kappa],$$

$$\varepsilon(t) \le A \int_0^t \varepsilon(s)\, ds + B_2 \sqrt[4]{h} \quad \text{for } t \in [\kappa, 2\kappa],$$

$$\varepsilon(t) \le B_2 e^{A\kappa} \sqrt[4]{h} \quad \text{for } t \in [\kappa, 2\kappa],$$

where $B_2 = B\sqrt[4]{h_0} + \sqrt{B}e^{A\kappa/2}\kappa$. By iterations, we obtain

$$\varepsilon(t) \le B_n e^{A\kappa}\, h^{2^{-n-1}} \quad \text{for } t \in [(n-1)\kappa, n\kappa].$$

This completes the proof.

4 A Continuous-time GARCH Model with State-dependent Delay

The following model for stock price x and its volatility $\sqrt{y}$ was derived in [1]:

$$(9) \quad \begin{aligned} dx(t) &= rx(t)\, dt + \sqrt{y(t)}x(t)\, dW(t), \\ \frac{dy(t)}{dt} &= \gamma V + \frac{\alpha}{\tau}\left\{\ln \frac{x(t)}{x(t-\tau)} - \mu\tau + \frac{1}{2}\int_{t-\tau}^{t} y(s)\, ds\right\}^2 - (\alpha+\gamma)y(t), \end{aligned}$$

where $\tau > 0$ is a constant. The model is derived from discrete-time GARCH model, and parameter estimation for S&P500 in [1] shows that the delay parameter varies considerably from year to year. This leads us to the assumption that τ is a function of state values. In this section, we assume $\tau = \tau(y(t-\kappa))$ so Theorem 3 can be applied.

From the previous section, the Euler dicrete-time scheme given below is convergent to the unique solution. The scheme is given by

(10)

$$\begin{aligned} x_{n+1} - x_n &= rx_n\Delta t + \sqrt{y_n}x_n\sqrt{\Delta t}\varepsilon_n, \\ \frac{y_{n+1} - y_n}{\Delta t} &= \gamma V + \frac{\alpha}{\tau(y_{n-k})}\left\{\ln \frac{x_n}{x_{n-N(y_{n-k})}} - \mu\tau(y_{n-k}) + \frac{\Delta t}{2}\sum_{i=0}^{N(y_{n-k})} y_{n-i}\right\}^2 \\ &\quad - (\alpha+\gamma)y_n, \end{aligned}$$

where $\tau(y_{n-k}) = \delta_0 + \hat{\tau}\exp(\rho y_{n-k})$ for some $-\rho, \hat{\tau}, \delta_0 > 0$, $N = [(\hat{\tau} + \delta_0)/\Delta t]$, $N(y_{n-k}) = [\tau(y_{n-k})/\Delta t]$, $k = [\kappa/\Delta t]$, $[\cdot]$ is the integer part and $\{\varepsilon_n\}_{n\ge 0}$

are i.i.d. Normal(0,1). Here, the initial data (x_n, y_n) is provided for $n = -N, \dots, 0$.

The particular choice of function τ is due to the following empirical observation. Since y represents volatility of the stock, the greater y the more volatile price, and therefore, the more active trading. Assuming the market's response to changes in the stock price is faster when the volatility is higher, we then conclude that the delay is a decreasing function of the volatility.

Let us try to find a fair price for the European call option written on the stock with maturity T and strike price K. It is known that the option price C is given by the following expectation:

$$C = E\left[e^{-rT}\max(x_T - K, 0)\right],$$

where r is risk-free interest rate and x_T is stock price at the time T. This expectation can be found using a Monte Carlo simulation of x_T approximated by the scheme (10).

Some simulation results are provided in the attached figures for different functions of state-dependent delay τ. They are presented as plots of implied volatility against strike price K. Note that implied volatility is computed using the inverse of Black-Scholes formula applied to simulated option price C.

It is well-known that the curve of the implied volatility of *market* option price has a U-shape, this is further confirmed by our plots. Observe also that the curvature of the graph is getting larger and larger when the value of the delay τ is increased. A constant delay cannot be used to control the height of the curve indepedently of the curvature, whereas varying delay can. Moreover, we provide some plots by using τ as an increasing or a periodic function to illustrate the variety of curves we can obtain. Solid lines represent 95%-confidence bounds for 10^6 simulations and dashed lines represent 95%-confidence bounds for 10^7 simulations.

References

[1] Y. Kazmerchuk, A. Swishchuk and J. Wu, Continuous-time GARCH model for stochastic volatility with delay, Preprint (2002).

[2] Y. Hu, S. Mohammed and F. Yan, Discrete-time approximations of stochastic differential systems with memory, Preprint (2001).

[3] I. Gyori, F. Hartung and J. Turi, Numerical approximations for a class of differential equations with time- and state-dependent delays, *Applied Math. Letters*, **8** (1995), 19-24.

[4] A. Skorohod, *Studies in the Theory of Random Processes*, Addison-Wesley Pub., Mass, 1965.

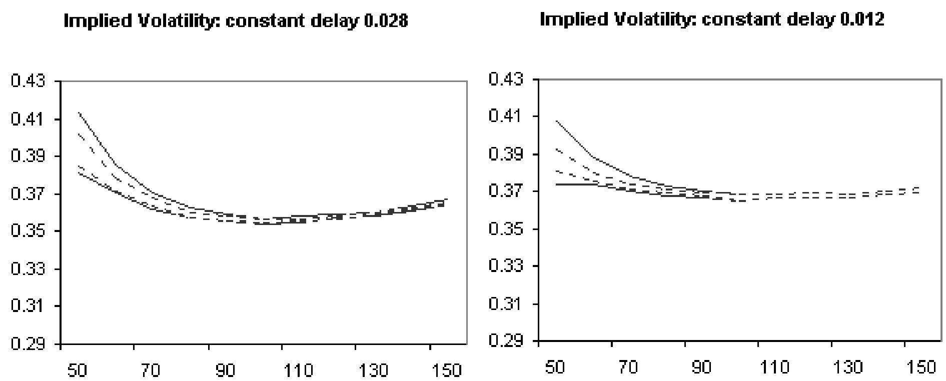

Figure 1: Implied volatility for models with constant delay

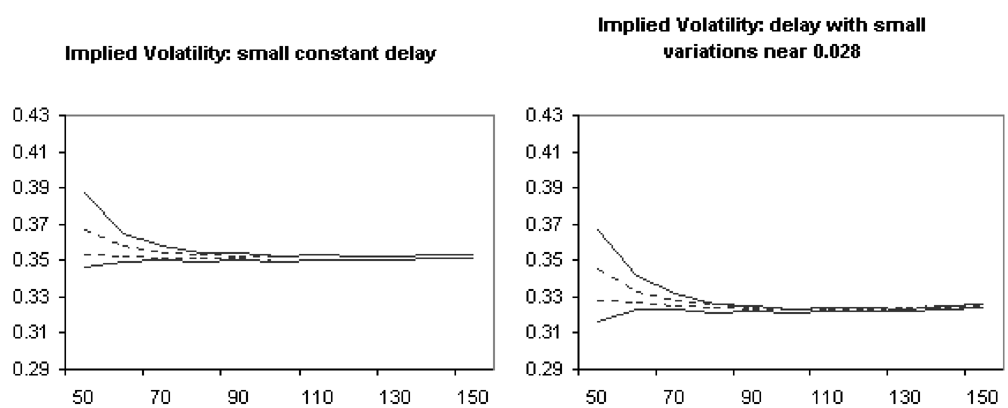

Figure 2: Implied volatility for models with nearly constant delay

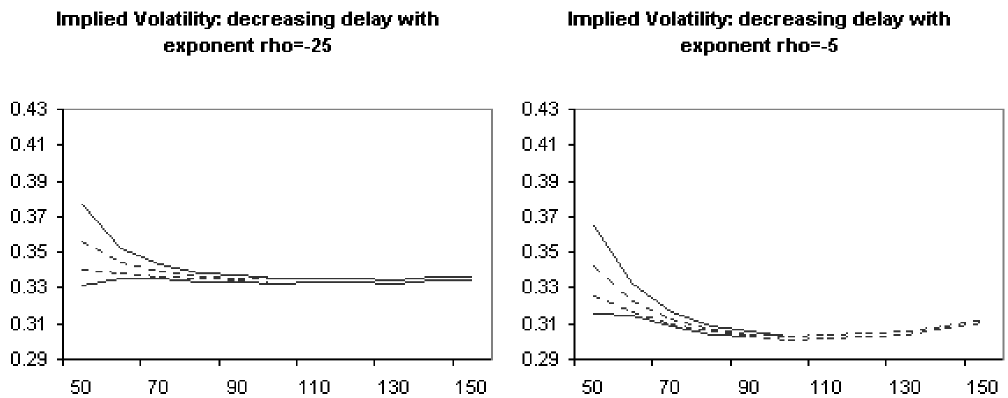

Figure 3: Implied volatility for models with decreasing state-dependent delay

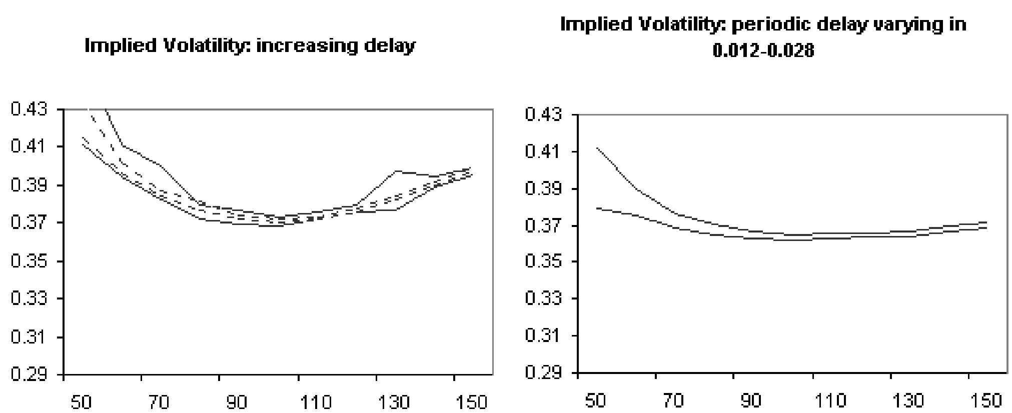

Figure 4: Implied volatility for models with state-dependent delays