Theory, Stochastic Stability and Applications of Stochastic Delay Differential Equations: a Survey of Results

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Abstract

This paper surveys some results in stochastic differential delay equations beginning with "On stationary solutions of a stochastic differential equations" by K. Ito and M. Nisio, 1964, and also some results in stochastic stability beginning with the "Stability of positive supermartingales" by R. Bucy, 1964. The problems discussed in this survey are the existence and uniqueness of solutions of stochastic differential delay equations (or stochastic differential functional equations, or stochastic affine hereditary systems), Markov property of solutions of SDDE's, stochastic stability, elements of ergodic theory, numerical approximation, parameter estimation, applications in biology and finance.

AMS (MOS) 2001 Subject Classifications: 34K50, 34D20, 34F05, 60H10, 60J60, 93E03, 62P05, 62P10.

1 Theory of Stochastic Delay Differential Equations

1.1. The first paper of this review on the SDDEs is [12] (Ito, Nisio, 1964).

Let $x(t)$, $t \in (-\infty, +\infty)$, be a stochastic process, $\mathcal{F}_{uv}(x)$ be a minimal σ-algebra (Borel), with respect to (w.r.t.) which $x(t)$ is measurable for every $t \in [u, v]$. Let $w(t)$ be a Wiener process, $t \in (-\infty, +\infty)$, $w(0) = 0$, and let $\mathcal{F}_{uv}(dw)$ be a minimal Borel σ-algebra, w.r.t. which $w(s) - w(t)$ is measurable for every (t, s) with $u \le t < s \le v$. Minimal σ-algebra, which contains $\mathcal{F}_1$, $\mathcal{F}_2$,..., is determined by $\mathcal{F}_1 \vee \mathcal{F}_2 \vee ... \vee \mathcal{F}_k$. Let s be arbitrary but fixed number.

A stochastic process

$$y(\theta) := x(s + \theta), \quad \theta \le 0, \tag{1.1}$$

is called a past (or memory) of the process x at the moment s and denoted by $\pi_s x$. C_- denotes a space of all continuous functions defined on the negative semiaxis $(-\infty, 0]$. C_- is a metric space with the metric

$$\rho_-(f, g) := \sum_{k=1}^{+\infty} 2^{-k} ||f - g||_k / (1 + ||f - g||_k), \tag{1.2}$$

where $||h||_k := \max_{-k \le t \le 0} |h(t)|$. Let $a(f)$ and $b(f)$ be continuous functionals defined on C_- with metric ρ_- given by (1.2).

Stochastic process $x(t)$ is called a solution of the SDDE

$$dx(t) = a(\pi_t x)dt + b(\pi_t x)dw(t), \tag{1.3}$$

$t \in (-\infty, +\infty)$, if

$$\mathcal{F}_{-\infty t}(x) \vee \mathcal{F}_{-\infty t}(dw) \tag{1.4}$$

is independent of $\mathcal{F}_{t+\infty}(dw)$ for every $t \in (-\infty, +\infty)$, and

$$x(t) - x(s) = \int_s^t a(\pi_\theta x)d\theta + \int_s^t b(\pi_\theta x)dw(\theta). \tag{1.5}$$

The solution $x(t)$ of equation (1.3), or equation (1.5) with property (1.4), is called stationary if $x(t)$ and $w(t)$ are stictly stationary dependent, i.e., if the distribution probabilities of the system

$$(x, dw) = (x(t); -\infty < t < +\infty, w(v) - w(u), -\infty < u < v < +\infty) \tag{1.6}$$

are invariant under shifts in time. A stationary solution, obviously, is a stationary process in the strong sense. Paper [12] mainly deals with stationary solutions. Consider the SDDE

$$x(t) = x(0) + \int_0^t a(\pi_s x)ds + \int_0^t b(\pi_s x)dw(s), \tag{1.7}$$

with the condition on the past

$$x(t) = x_-(t), \quad t \leq 0, \tag{1.8}$$

where $x_-(t)$, $t \leq 0$, is a given continuous process. Since a solution of this equation satisfies (1.3) for $t \geq 0$, the solution is also called a one-sided solution.

The authors prove:

1) the existence of one-sided solutions provided the growth of the coefficients $a(f)$ and $b(f)$ is linear at the most; this restriction on the growth prevents the "exit" of solutions to infinity during a finite time; the authors use the theory of completely bounded sets of stochastic processes by Prohorov-Skorokhod;

2) the existence of stationary solutions in the sense of system (1.6) in terms of one-sided solutions;

3) a complete description of possible stationary solutions of SDEs of Markov type (i.e., equations with coefficients a and b dependent on the present value $f(0)$ only and satisfying a Lipschits condition).

Let $(\Omega, \mathcal{F}, P)$ be a probability space with the past process $x_-(t)$, $t \leq 0$ and Winer process $w(t)$, $t \in (-\infty, +\infty)$t. Without loosing a generality we assume, that $\mathcal{F}_{-\infty 0}(x_-) \vee \mathcal{F}(dw) = \mathcal{F}$.

Let the following conditions be satisfied for equations (1.7)-(1.8):

A.1 $a(f)$ and $b(f)$ are continuous on $C_-(\rho_-)$;

A.2 there exist a constant $M_1 > 0$ and a bounded measure dK_1 on $(-\infty, 0]$ such that

$$|a(f)| + |b(f)| \leq M_1 + \int_{-\infty}^0 |f(t)|dK_1(t);$$

A.3 $E(x_-(t))^4 \leq C$, $t \leq 0$, for some constant C;
A.4 $\mathcal{F}_{-\infty 0}(x_-)$ is independent of $\mathcal{F}_{0+\infty}(dw)$.

Condition A.4 necessarily follows from the assumption that

$$\mathcal{F}_{-\infty t}(x) \vee \mathcal{F}_{0t}(dw) \tag{1.9}$$

is independent of $\mathcal{F}_{t+\infty}(dw)$ for every $t \geq 0$.

Theorem 1 *(Existence of one-sided solution) Assume conditions A.1-A.4. Then there exists a solution $x(t)$ of equation (1.7) satisfying (1.8)-(1.9) and such that $E|x(t)|^4 \leq \gamma \exp(\gamma t), \gamma > 0$.*

The proof is based on two major steps.

The first step is to construct approximated polygonal solutions. For $h > 0$ define a Cauchy polygonal solution by $x_h(t) := x_-(t)$, if $t \leq 0$, and $x_h(t) := x_h(nh) + a(\pi_{nh} x_h)(t - nh) + b(\pi_{nh} x_h)(w(t) - w(nh))$, if $nh \leq t \leq (n+1)h$, $n = 0, 1, 2, ...$ Introduce a step function by $\phi_h(t) := nh$, if $nh \leq t \leq (n+1)h$, $n = 0, 1, 2, ...$ Then $x_h(t)$ satisfies the equation

$$x_h(t) = x_h(0) + \int_0^t a(\pi_{\phi_h(s)} x_h) ds + \int_0^t b(\pi_{\phi_h(s)} x_h) dw(s). \tag{1.10}$$

It's proved in the paper that

1) $c_h(t) := \sup_{s \leq t} E|x_h(s)|^4 < \gamma \exp(\gamma t)$, with $\gamma := \gamma(M, dK, C)$, which is independent on h;

2) $E((x_h(t) - x_h(s))^4 \leq \gamma_n |t-s|^{3/2}$, $0 \leq s < t \leq h$, $n = 1, 2, 3, ...$, $\gamma_n := \gamma(n, M, ||K||, C)$;

3) $\{x_h; h > 0\}$ in (1.10) is completely L-bounded.

The second step is to construct a solution. There exists a L-Cauchy sequence such that $(x_{h(u)}, B, x_-) \to (y_{+\infty}, B_{+\infty}, x_-)$ as $h(n) \to 0$ and

$$y_\infty(t) = y_\infty(0) + \int_0^t a(\pi_s y_\infty) ds + \int_0^t b(\pi_s y_\infty) dB_{\infty(s)} \tag{1.11}$$

with probability (w.p.) 1. The inequality $E(y_\infty(t))^4 \leq \gamma \exp(\gamma t)$ holds and $x(t) = y_\infty(t)$ is a solution of equation (1.7). To prove (1.11) it's sufficient to prove it for any $t \geq 0$ w.p.1. Let

$$I_n := \int_0^t a(\pi_s y_n) ds + \int_0^t b(\pi_s y_n) dB_n(s) \tag{1.12}$$

Then using the step function $\phi_h(t)$ for I_n in (1.12) we obtain

$$\begin{aligned} I_n &= \sum_{k=0}^{m-1} a(\pi_{kh} y_h) h + a(\pi_{mh} y_h)(t - mh) + \\ &+ \sum_{k=0}^{m-1} b(\pi_{kh} y_h)[w((k+1)h) - w(kh)] + b(\pi_{mh} y_h)[w(t) - w(mh)] \end{aligned} \tag{1.13}$$

where $mh \le t < (m+1)h, n = 1, 2, 3, \dots$. Since $P(\rho(y_n, y_\infty) \to 0) = 1$ the sequence $\{y_n; n = 1, 2, ..\}$ is completely L-bounded, and $P(|I_n(h) - I_n| > \epsilon)) < \epsilon$, then $P(|y_\infty(t) - y_\infty(0) - I_\infty| > 6\epsilon) < 3\epsilon$ (see (1.11)-(1.14)). Finally, $E(y_\infty(t))^4 \le \lim_{n\to+\infty} E(y_n(t))^4 = \lim_{n\to+\infty} E(x_{h(n)}(t))^4 \le \gamma \exp(\gamma t)$.

Theorem 2 *If one-sided solution has a fourth moment $E(x(t))^4 \le \alpha$, $t \ge 0$, $\alpha \in (0, +\infty)$, then there exists a stationary solution of equation (1.7).*

The idea of the proof is as follows.

Let $x_s(t) := x(s+t)$ and $w_s(t) := w(s+t) - w(s)$. Then

$$dx_s(t) = a(\pi_t x_s)dt + b(\pi_t x_s)dw_s(t), \tag{1.14}$$

$t \ge -s$, $x_s(t) = 0$, $t \le -s$, $w_s(w_s(0) = 0)$ is a Wiener process.

Let P_s be a probability distribution of the (x_s, B_s)-measure on $C^2 = C \times C$, and θ_s be a shift operator on C^2: $(\tilde{f}, \tilde{g}) := \theta_s(f, g)$, i.e., $\tilde{f}(t) = f(s+t)$, $\tilde{g}(t) = g(t+s)$. Since $(x_s, B_s) = \theta_s(x, B) = \theta_s(x_0, B_0)$, then

$$P_s(E) = P_0(\theta_{-s}E), \tag{1.15}$$

where x_s is defined by (1.14). Since $\pi_s f$ is continuous for $(s, f) \in (-\infty, +\infty) \times C$, then $P_s(E)$ is Borel measurable in s for any $E \in \mathcal{B}(C^2)$. Because of this $Q_T(E) := 1/T \int_0^T P_s(E)ds$ is a probability measure on $(C^2, \mathcal{B}(\mathcal{C}^\in)$. (w_s) is completely L-bounded: $E((w_s(v) - w_s(u)))^4 = 3(u-v)^2 \le 3\sqrt{2n}|v-u|^{3/2}$ if $|u|, |v| \le n$. $(x_s, s \ge 0)$ is also completely L-bounded. Hence, (x_s, B_s) is L-bounded and there exists a compact set $K = K(\epsilon) \in C^2$ such that $P_s(K) > 1 - \epsilon$ and $Q_T(K) > 1 - \epsilon$, i.e., $(Q_T; T > 0)$ is L-bounded, where P_s is defined by (1.15). Hence, there exists a sequence $T_n \to +\infty$ such that $Q_{T_n} \to Q$, where Q is a measure on C^2. Let $(\tilde{x}, \tilde{w})$ be C^2-valued real variable with distribution Q. Hence $\tilde{w}(t)$ is a Wiener process, $\tilde{w}(0) = 0$; $\tilde{x}$ is strictly stationary connected with $d\tilde{w}(t)$ and $d\tilde{x}(t) = a(\pi_t\tilde{x})dt + b(\pi_t\tilde{x})d\tilde{w}(t)$, and $\tilde{x}(t)$ is a stationary solution.

Assume that the functionals

$$a(f) = a_0(f)f(0) + a_1(f); b(f) = b_0(f)f(0) + b_1(f) \tag{1.16}$$

$a_i(f)$, $b_i(f)$, $i = 0, 1$, are bounded and continuous in $f \in C_-$. In order for a stationary solution of equation (1.7) to exist it's sufficient that there exists $m > 0$ such that $2a_0(f) + b_0^2(f) \le -m$ for all $f \in C_-$.

Let

$$a(f) = -cf(0) + a_1(f), \quad c > 0, \tag{1.17}$$

where $a_1(f)$, $b(f)$ are measurable for $f \in C_-$ and such that

$$|a_1(f) - a_1(g)|^2 \le \int_{-\infty}^0 |f(t) - g(t)|^2 dK_1(t),$$

$$|b(f) - b(g)|^2 \le \int_{-\infty}^0 |f(t) - g(t)|^2 dK^2(t),$$

and

$$c > ||K_1||^{1/2} + 1/4||K^2|| + 1/4(||K_2||^2 + 8||K_1||^{1/2}||K_2||)^{1/2}.$$

Then a stationary solution of equation (1.7) with the functional $a(f)$ given by (1.17) exists and $E(x(0))^2 < +\infty$.

Let $a(f)$ and $b(f)$ in (1.16) be linear functions given by

$$a(f) = M_1 + \int_{-\infty}^{0} f(t)dK_1(t), b(f) = M_2 + \int_{-\infty}^{0} f(t)dK_2(t). \tag{1.18}$$

Then $x(t)$ is a stationary solution of (1.7). If $a(f)$ and $b(f)$ are of the form (1.18) and $-c$ is a jump of dK_1 at 0 then a stationary solution of (1.7) exists in the following two cases:

1) $c > c_1 + 1/2c_2^2$;

2) $c > c_1 + 1/4c_2^2 + 1/4(c_2^4 + 8c_1c_2^2)^{1/2}$,

where $c_1 := \int_{-\infty}^{0} |dK_1(t)|, c_2 := \int_{-\infty}^{0} |dK_2(t)|$,

Let $a(f) = b(f) = 1$, and

$$dx(t) = [\mu + \int_{-\infty}^{0} x(t+s)dK(s)]dt + dw(t). \tag{1.19}$$

Equation (1.19) is solved by using generating functions and spectral theory of operators. If $a(f) = \alpha(f(0))$, $b(f) = \beta(f(0))$, i.e. the coefficients depend only on $f(0)$, then one obtains an ordinary SDE and $x(t)$ is a well-known diffusion process.

1.2. In paper [20] (Fleming, Nisio, 1966) on SDDEs the authors apply the above mentioned results to stochastic control problems for the equation

$$dx(t) = \alpha(t, \pi_t x)dU(t) + \beta(t, \pi_t x, \pi_t w)dw(t), t \geq 0, \tag{1.20}$$

with the past condition x_-. If $x(t) = x_-(t)$, $t \leq 0$, and $\mathcal{F}_{-\infty t}(x) \vee \mathcal{F}_{0t}(U) \vee \mathcal{F}_{-\infty t}(w)$ is independent of $\mathcal{F}_{t+\infty}(dw)$ for all $t \geq 0$, $U(t)$ is a control process satisfying $|U_i(t) - U_i(s)| \leq |t-s|; U(0) = 0$. The paper deals with the existence and uniqueness of solutions of equation (1.20) and its optimal stochastic control.

1.3. The next paper of this review is [27] (Kushner, 1968).

Let C be the space of continuous functions on the interval $[-r, 0]$, $r > 0$, and let $x(t)$ be a vector-valued stochastic process. Define the process $x_t \in C$ by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$. Let $|x(t)|^2 := \sum_{i=1}^n x_i^2(t)$ and $||x_t|| := |x(t+\theta)|$, $\theta \in [-r, 0]$.

Let $x(t)$ satisfy the vector SDDE

$$x(t) = x(0) + \int_0^t f(x_s)ds + \int_0^t g(x_s)dw(s), \tag{1.21}$$

where x_0, f and g are given functions, $w(s)$ is a vector-valued normalized Wiener process with independent components. Equations of the type (1.21) have been studied in [12] and [20].

Let $|f|^2 := \sum_i f_i^2, |g|^2 := \sum_{i,j} g_{ij}^2$. Assume the vector f and matrix g satisfy the following conditions

A1) f_i and g_{ij} are continuous real-valued functions on C;

A2) $x(t)$ is continuous w.p.1 in the interval $[-r, 0]$ and independent of $w(s) - w(0)$, $s \geq 0$, and $E|x(t)|^4 < +\infty$;

A3) there exists $M < +\infty$ and bounded measure μ on $[-r, 0]$ such that

$$|f(\phi) - f(\psi)| + |g(\phi) - g(\psi)| \leq \int_{-r}^{0} |\phi(\theta) - \psi(\theta)| d\mu(\theta), \quad \forall \phi, \psi \in C \qquad (1.22)$$

with $|f(0)| + |g(0)| \leq M$.

Condition A3) implies

A3$'$) there exists $M < +\infty$ and bounded measure μ on $[-r, 0]$: with $|f(0) + |g(0)| \leq M$ and

$$|f(\phi) - f(\psi)|^2 + |g(\phi) - g(\psi)|^2 \leq \int_{-r}^{0} |\phi(\theta) - \psi(\theta)|^2 d\mu(\theta) \qquad (1.23)$$

Theorem 3 *Assume A1) and A3). Let $x(t)$ and $y(t)$ be solutions of equation (1.21) corresponding to the initial conditions $x_0 = x$ and $y_0 = y$, respectively, where x and y satisfy A2). Then*

$$E \max_{T \geq t \geq 0} |x(t) - y(t)|^2 \leq KE|x(0) - y(0)| + \int_{-r}^{0} E|x(\theta) - y(\theta)|^2 d\mu(\theta), \qquad (1.24)$$

where K depends on T, μ and M of A3) only. The solution of equation (1.21) is unique in the sense that if $x = x_0$ satisfies A2) then two solutions with bounded second moments must coinside w.p.1 with respect to the inequalities (1.23) and (1.24).

Theorem 4 *Assume A1) - A3). Then*

$$E \max_{T \geq t \geq 0} |x(t) - x(0)|^2 \leq KTE\{1 + \int_{-r}^{0} (|x(\theta)|^2 + |x(\theta) - x(0)|^2) d\mu(\theta)\},$$

and with $x_0 \in C$ fixed one has

$$|Ex(h) - x(0) - hf(x_0)| = o(h), \quad |E(x(h) - x(0))(x(h) - x(0))' - hg(x_0)g'(x_0)| = o(h).$$

Theorems 3 and 4 are used then in the paper to establish the stochastic continuity, the strong Markov character of x_t, and some characterizations of the weak infinitesimal operator of x_t.

Theorem 5 *Assume A1) - A3) and let $x_0 = x \in C$. Then x_t is a continuous strong Markov process on the topological state space $(C, \mathcal{C}, \mathcal{B})$ with the killing time $\xi(\omega) = +\infty$ w.p.1, where $\mathcal{C}$ is the family of open sets in C and $\mathcal{B}$ is the Borel field over $\mathcal{C}$.*

Definition. A real-valued function F on C is said to be in the domain of $\tilde{A}$, the weak infinitesimal operator, if the limits

$$\lim_{t\to 0}(E_x F(x_t) - F(x))/t = q(x), \tag{1.25}$$

$$\lim_{t\to 0} E_x q(x_t) = q(x)$$

exist pointwise in C and the sequence is uniformly bounded in x. We define then $q(x) := \tilde{A}F(x)$ and use the notation $\tilde{A}_R$ for the weak infinitesimal operator of the process $\tilde{x}_t = x_t$ that stopped at time $\tau := \inf\{t : x_t \notin R\}$, where R is an open set.

Lemma 1 *Assume A1), A2) and the following condition:*

A4) For all $\rho > 0$ there exists a bounded measure μ_ρ on $[-r, 0]$ such that inequality (1.22) is satisfied for $||\psi|| \leq \rho$ and $||\phi|| \leq \rho$ (with μ_ρ replacing μ) and the condition $|f(0)| + |g(0)| \leq M < +\infty$ holds for equation (1.21).

Let $\tilde{A}$ be the weak infinitesimal operator of the process $\tilde{x}(t)$ that solves equation (1.21) (with $\tilde{f}$, $\tilde{g}$ replacing f, g respectively) and that satisfies conditions A1)-A3). Assume also that $\tilde{f} = f$, $\tilde{g} = g$ on the bounded open set R. Then $\tilde{f}$ and $\tilde{g}$ can be defined outside R so that $||\tilde{x}_t|| \leq K < +\infty$. Let F be continuous and bounded on bounded sets.

Then if $F \in \mathcal{D}(\tilde{A})$ and $\tilde{A}F = q$ is bounded on bounded sets, the restriction of F to R is in $\mathcal{D}(\tilde{A}_R)$ and $\tilde{A}F = \tilde{A}_R F$ on R.

Theorem 6 *Assume A1), A2) and A4), and $x_0 = x \in C$. Let $F(x) = G(x(0))$ have continuous second derivatives w.r.t. $x(0)$. Then $F(x) \in \mathcal{D}(\tilde{A}_R)$ and*

$$\tilde{A}_R F(x) = LG(x(0)) = q(x) = G'_u(x(0))f(x) + 1/2 \sum_{i,j} G''_{u_I u_j}(x(0))\sigma_{ij}(x), \tag{1.26}$$

where $\sigma_{ij} = \sum_k g_{ik} g_{kj}$.

It's not simple to completely characterize the domain of the weak infinitesimal operators A_R or $\tilde{A}_R$ in (1.26) of the processes x_t and $\tilde{x}_t$ (see (1.25)). For example, $F(x) = x(-a)$, $r > a > 0$, is not necessarily in $\mathcal{D}(\tilde{A})$, since $x(t)$ is not necessarily differentiable. Basically it is possible to study functions $F(x)$ whose dependence on $x(\theta), -r \leq \theta \leq 0$, is in the form of an integral. The dependence of $F(x)$ on $x(0)$ can be more arbitrary (see Theorem 6).

Theorem 7 *Assume the conditions of Theorem 6 with function F given by*

$$F(x) = \int_{-r}^{0} h(\theta) H(x(\theta), x(0)) d\theta. \tag{1.27}$$

Let h be defined and have the continuous derivative on an open set containing $[-r, 0]$. Let $H(\alpha, \beta)$, $H_{\beta_i}(\alpha, \beta)$ and $H_{\beta_i\beta_j}(\alpha, \beta)$ be continuous in α and β. Then $F(x) \in \mathcal{D}(\tilde{A}_R)$ and

$$\begin{aligned} \tilde{A}_R F(x) = q(x) &= h(0)H(x(0), x(0)) - h(-r)H(x(-r), x(0)) - \\ &- \int_{-r}^{0} h_\theta(\theta) H(x(\theta), x(0)) d\theta + \int_{-r}^{0} h(\theta) LH(x(\theta), x(0)) d\theta, \end{aligned} \tag{1.28}$$

where the operator L is defined by (1.26) and acts on H as a function of $x(0)$ only, function $F(x)$ is defined in (1.27).

Theorem 8 *Let G be a twice continuously-differentiable real-valued function of a real argument. Assume the conditions of Theorem 7. Then $F_1(x) = G(F(x)) \in \mathcal{D}(\tilde{A}_R)$ and*

$$\tilde{A}_R F_1(x) = G'_F(F(x))\tilde{A}_R F(x) + 1/2 G''_{FF}(F(x))B, \tag{1.29}$$

where

$$B = \int_{-r}^{0}\int_{-r}^{0} h(\theta)h(\rho)\sum_{i,j} H_{\beta_i}(x(\theta), x(0))H_{\beta_j}(x(\rho), x(0))\sigma_{ij}(x)d\theta d\rho.$$

Theorem 8 is an extension of Theorems 6 and 7 with operator $\tilde{A}_R$ in (1.29) in place of operators $\tilde{A}_R$ in (1.26) and (1.28). Their proofs are straightforward computations.

Corollary. Let $F^a(\beta)$ and $F^b(\alpha,\beta)$ satisfy the hypotheses of Theorems 6 and 7, respectively. If G is twice continuously differentiable then $F_1 = G(F^a(x) + F^b(x)) \in \mathcal{D}(\tilde{A}_R)$ and

$$\tilde{A}_R F_1(x) = G_F(F^{a(x)} + F^b(x))(\tilde{A}_R F^a(x) + \tilde{A}_R F^b(x)) + 1/2 G_{FF}(F^a(x) + F^b(x))B, \tag{1.30}$$

where B and $C_i(x)$ are given by

$$B = \sum_{ij} \sigma_{ij}(x)[F^a_{\beta_i}(x(0)) + C_1(x)][F^b_{\beta_j}(x(0)) + C_j(x)],$$

$$C_i(x) = \int_{-r}^{0} h(\theta)H_{\beta_i}(x(\theta), x(0))d\theta.$$

$F^a_{\beta_i}$ in (1.30) and H_{β_i} are the derivatives w.r.t. the i-th component of $x(0)$. The above mentioned results of [27] are used to derive stability theorems for equation (1.21).

1.4. Consider a representation theorem from [44] (Chojnowska-Michalik, 1978) for general stochastic delay equations.

Fix $h \in [0, +\infty)$ and let $G = L^2([-h, 0], R^m)$; Set $\mathcal{H} = H \times G$ with elements denoted by

$$\phi = \begin{cases} \phi(0) \in H \\ \phi(\cdot) \in G. \end{cases}$$

Let $\mu : [-h, 0] \to \mathcal{L}(R^n, R^n)$ be a function of bounded variation. Some results of [33] enable one to consider the linear delay equation

$$dx(t) = \left(\int_{-h}^{0} d\mu(\theta)x(t+\theta)\right) dt,$$

in the semigroup context in the state space $\mathcal{H}$. The corresponding semigroup F_t describes the evolution of the solution segments

$$\begin{cases} x(t), \\ x(t+\theta), \quad \theta \in [-h, 0], \end{cases}$$

and the following operator

$$\mathcal{A}\phi = \begin{cases} \int_{-h}^{0} d\mu(\theta)\phi(\theta) \\ d\phi/d\theta, \end{cases}$$

with $\mathcal{D}(\tilde{\mathcal{A}}) = W^{1,2}([-h, 0]; R^n)$ is the generator of F_t under some additional assumptions (see [39]). The following form of function μ

$$\mu(u) = \int_{-h}^{u} L(\theta)d\theta + \sum_{i=1}^{p} L_i\delta_{\theta_i}([-h, u]),$$

where δ_{θ_i} stands for the Dirac measure, is sufficient for $\mathcal{A}$ to be the generator of the semigroup F_t

Theorem 3.1 of [44] shows that a solution of the stochastic linear delay equation in R^n with the state dependent martingale noise

$$dz_t = (\int_{-h}^{0} \mu(\theta)z(t+\theta))dt + d\mathcal{M}(z)_t,$$

is exactly the R^n-vector of the mild solution of the following stochastic evolution equation in $\mathcal{H}$

$$dy_t = \mathcal{A}y_t dt + d\tilde{M}(y)_t. \tag{1.31}$$

Moreover, it's shown that equation (1.31) can have no strong solutions (Example 3.6 of the paper).

A similar but weaker representation result (in the case of the noise being a Wiener process) was obtained in [39] in a different more difficult way. The proof makes use of the semigroup approach to boundary value broblems of [40] and of some results on stochastic evolution equations. The authors' theorem enables them to make solutions of stochastic delay equations Markovian as well as to apply certain results on infinite dimensional stability [40], such as filtering and control.

1.5. Consider a qualitative behavior of stochastic delay equations from [59] (Scheutzow, 1984) with a bounded memory of the form

$$dx(t) = F(x_t)dt + dw(t). \tag{1.32}$$

Here $w(t)$, $t \geq 0$, is a d-dimentional Wiener process, $F : C([0, 1], R^n) \to R^n$ is measurable function w.r.t. the Borel σ-algebra $\mathcal{B}$ induced by the norm $||f|| := \sup_{s\in[0,1]} ||f(s)||_2$ on $C = C([0, 1], R^n)$, and C is the space of continuous R^n-valued functions on the interval $[0, 1](|| \cdot ||_2$ stands for the l_2-norm in $R^n)$.

For any R^n-valued stochastic process y, let y_t be defined by $y_t(s) := y(t - 1 + s)$, $s \in [0, 1]$.

The author assumes that F is locally bounded (bounded memory), i.e., for all $R > 0$: $C_R := \sup_{f\in C} ||F(f)||_2 < +\infty$.

A dichotomy is proved concerning the recurrence properties of solutions of equation (1.32). If the solution process is recurrent, there exists an invariant measure π on the state space C which is unique. The author states a sufficient condition for the existence of an invariant probability measure in terms of Lyapunov functionals and gives two examples, one being the stochastic-delay version of the famous logistic equation of population growth, namely:

$$dz(t) = (k_1 - k_2 z^{k_3}(t-1))z(t)dt + k_4 z(t)dw(t), \tag{1.33}$$

where $k_i > 0, i = \overline{1,4}$.

Equation (1.33) ties the relative growth $dz(t)/dt(1/z(t))$ of a population to the birth rate k_1 and a negative feedback term appearing due to limitation of resourses (which is a monotone function of the population). It seems to be reasonable to assume, as it is done in many similar deterministic models, that the feedback in the system is with a time delay. Furthermore, the author allows the relative growth be perturbed by a white noise. Equation (1.33) is frequently considered and sdudied in the biomathematical literature (see e.g. [36],[49]). Usually it's assumed that $k_3 = 1$ and either $k_4 = 0$ or there is no time delay. For $k_4 = 0$, $k_3 = 1$ and suitable k_1 and k_2 it's known [36] that apart from the constant solution $z = k_1/k_2$ equation (1.33) has a periodic solution which is robust against certain perturbations. The author proves that the periodic solution of (1.33) is robust against stochastic perturbations. Equation (1.33) is not of the type equation (1.32) is. By using the transformation $x(t) = 1/k_4 \log z(t) + (logk_2/k_4)(k_3k_4)^{-1}$ and Ito formula it is reduced to the form

$$dx(t) = (k_1/k_4 - k_4/2 - \exp(k_3k_4x(t-1)))dt + dw(t),$$

provided $z(t) > 0$ and $k_4 > 0$. Finally, the author studies approximations of delay equations by Markov chains.

1.6. In paper [74] (Mohammed, Scheutzow, 1990) the authors study Lyapunov exponents and stationary solutions for affine stochastic delay equations of the following kind

$$x(t) = \nu(0) + \int_0^t \int_J d\mu(u)x(s+u)ds + M(t),\ t \geq 0, \quad x(u) = \nu(u),\ -r \leq u \leq 0, \tag{1.34}$$

where $M(t)$ is a (possibly random) R^n-valued function which is locally Lebesque-integrable and satisfies $M(0) = 0$. Let $r > 0$, $J = [-r, 0]$, and take the initial state η to be in $\tilde{C}$, the linear space of Lebesque-integrable R^n-valued functions on J. μI is a function of bounded variation on J (or finite signed measure on J). Under a solution of equation (1.34) one means a function $x(t)$, $t \geq -r$, which is locally integrable, satisfies (1.34) $\forall t \geq 0$, and $x(u) = \nu(u)$, $-r \leq u \leq 0$.

Two cases for the forcing function M are treated in the paper:

M is locally integrable deterministic, or

M is a random process with stationary increments.

The Lyapunov spectrum of the homogeneous ($M = 0$) equation (1.34) is used to decompose the state space into a sum of a finite-dimensional and a infinite-dimensional subspaces. If the homogeneous equation is hyperbolic and M has stationary increments, the existence and uniqueness of a stationary solution for the equation is proved.

The existence of Lyapunov exponents for the affine equations and their dependence on the initial conditions are also studied.

Let $\tilde{B} \subset \tilde{C}$ be the subspace of bounded Borel measurable functions ν on J with the norm $||\nu|| := \sup_{u\in J} |\nu(u)|$, where $|\cdot|$ is the usual Euclidian norm in R^n. Set $H : \tilde{B} \to R^n$; $H\nu := \int_J d\mu(u)\nu(u)$, $\nu \in \tilde{B}$.

Then equation (1.34) can be rewritten as the following affine SDFE

$$dx(t) = Hx_t dt + dM(t). \tag{1.35}$$

Equations of the similar to (1.34) and (1.35) types have been studied in [61, 59, 12, 60]. Since equation (1.34) involves no stochastic integral it seems natural to treat the equation pathwise even if M and/or ν are random. In [65] and [61] a special case where $M(t)$ is a Wiener process was treated and a stochastic approach is used. An equations with a linear in $x(t)$ coefficient in front of $dM(t)$ in (1.35) have also been studied. For the delay case see [73].

1.7. Paper [76] (Mohammed, 1992) contains a review of known results and also discusses new ones concerning the existence of flows and the characterization of Lyapunov exponents for trajectories of stochastic linear and affine hereditary systems. Such systems (also called SDFEs) are SDEs in which the differential of the state variable x depends on its current value $x(t)$ at time t as well as its previous values $x(s)$, $t-r \leq s \leq t$. The author deals almost exclusively with the finite history case: $0 \leq r < +\infty$.

Consider the following stochastic affine hereditary system

$$dx(t) = \sum_{i=0}^{m} [\int_{-r}^{0} \nu_i(t)(ds)x(t+s)]dz_i(t) + dQ(t), t > 0, \tag{1.36}$$

with $x(0) = v$ and $x(s) = \eta(s)$ for $-r < s < 0$. The above system is considered on the complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, vectors in R^n (or C^n) are column vectors with the standard Euclidean norm $|\cdot|$. The noise in (1.36) is provided by $\mathcal{F}_t$-semimartingale $z_i : R^+ \times \Omega \to R$, $i = 0, 1, 2, ..., m$; $Q : R^+ \times \Omega \to R^n$ with jointly stationary increments. The memory is prescribed by stationary $\mathcal{F}_t$-adapted measure-valued processes ν_i, $i = \overline{0, m}$, such that each $\nu_i(t, \omega)$ is an $n \times n$-matrix-valued measure on $[-r, 0]$. The solution $x : [-r, 0] \times \Omega \to R^n$ is a measurable $\mathcal{F}_t$-adapted process with $x|(0, +\infty) \times \Omega$ having a.a. right-continuous with left limits sample paths (so called "cadlag"). The initial condition is a (possibly random) pair $(v, \nu) \in R^n \times \Xi$, where Ξ is some Banach space containing all cadlag paths $[-r, 0] \to B^n$, e.g., $\Xi = C([-r, 0], R^n)$, $D([-r, 0], R^n)$, $L^2([-r, 0], R^n)$, or a weighted L^2 space $L^2_\rho((-r, 0], R^n$, so as to allow for the infinite fading memory case $r = +\infty$ (see [60, 21]). In order to observe the dynamics of (1.36) it's convinient to define the segment $x_t \in \Xi$ by $x_t(., \omega)(s) := x(t+s, \omega)$, $t \geq 0$, $-r \leq s \leq 0$.

In the deterministic case when $Q = 0$, $\nu_i(t, \omega)$ are fixed in (t, ω), and $z_i(t) = t$, $i = \overline{0, m}$, almost surely, the idea to use the second Lyapunov method to study the stability of this SDDE goes back to [2], pp 126-175. In this case the existence of solutions and the asymptotic stability of trajectories $x_t \in \Xi = C([-r, -], R^n)$ were studied extensively by J. Hale and his collaborators in the sixties [16, 42, 2, 22, 9] and by others.

The corresponding issues in the case $\Xi = L^2([-r, 0], R^n)$ were studied in [34] for the finite memory case (see also [50] and the references therein for systems with infinite memory).

For the stochastic hereditary white-noise case ($z_0(t) = t$, $z_i(t)$ with $Q(t)$ an independent Brownian motion and $\nu_i(t)$ fixed, the existence of the $(\mathcal{F}_t)_{t\geq 0}$ adapted solutions and their asymptotic stability were considered by several authors, see e.g. [12, 27, 61, 67, 69, 60, 65, 59, 66].

Extensions of the existence results to the case of semimartingale noises z_i, Q, were discussed in [43]. The author's discussion focuses on results concerning almost everywhere asymptotic stability of the trajectory $(x(t), x_t) \in \mathcal{E} := R^n \times \Xi$ of the stochastic hereditary system (1.36). The following issues are discussed:

(i) Existence of measurable stochastic semi-flows $X : R^+ \times \Omega \times \mathcal{E} \to \mathcal{E}$ for (1.36) with the properties:
a) if x is the solution of (1.36) with initial data $(v, \nu) \in \mathcal{E}$, then $x(t, \cdot, (v, \nu)) = (x(t), x_t)$ for $\forall t \geq 0$, a.s.;
b) each map $x(t, \omega, .), t \in R^+$, $\omega \in \Omega$, is continuous affine linear operator on $\mathcal{E}$;

(ii) A characterization of the almost sure Lyapunov exponents

$$\limsup_{t \to +\infty} 1/t \log ||(x(t), x_t)||_{\mathcal{E}}$$

for a given natural norm on the space state $\mathcal{E}$. For example, if $\Xi = L^2([-r, 0], R^n)$, one usually takes the Hilbert norm: $||(v, \nu)||^2_{M_2} := |v|^2 + \int_{-r}^0 |\nu(s)|^2 ds$, $(v, \nu) \in R^n \times L^2([-r, 0], R^2)$, on the classical Delfour-Mitter space $\mathcal{E} := M_2 := R^n \times L^2([-r, 0], R^n)$ (see [33]);

iii) A study of the hyperbolicity of equation (1.36) for the case of non-zero Lyapunov exponents. This is of interest for two reasons. In the linear case ($Q = 0$), the hyperbolicity leads to an exponencial dichotomy with a flow-invariant saddle-point splitting. When $z_i(y) = t$ with $\nu_i(t)$ fixed, $1 \leq i \leq m$, Q having stationary increments and (1.36) being hyperbolic, it turns out that the affine hereditary equation admits a unique stationary solution.

1.8. In the survey paper [82] (Mandrekar, 1994) the author considers delay system of the form

$$dx_t = \int_{-r}^0 dN(s) x_{t+s} + B(x_t) dw(t) \tag{1.37}$$

with $x_t = 0, t < 0$, where $x_t \in R^n$ and $N(\cdot)$ is a continuous from the left function of bounded variation on $[-r, 0]$ with values in the set of $n \times n$ matrixes (see also [46, 48]).

The author derives a result on the stability of solutions of the stochastic delay differential equation (1.37) with $B(x)$ being a nonlinear matrix-valued function and W being a Brownian motion.

1.9. In book [92] (Da Prato, Zabczyk, 1996), Chapter 10 "Stochastic delay systems", the authors study invariant measures for SDDEs by the dissipative method. Let $d, m \in N$, $r > 0$, $a(\cdot)$ be a finite $d \times d$-matrix-valued measure on $[-r, 0]$ and b be a $d \times d$ matrix, f be a mapping from R^d into R^d, and $W(\cdot)$ be a standard m-dimentional Wiener process.

In the chapter the authors deal with a SDDE of the form

$$\begin{aligned} dy(t) &= \left(\int_{-r}^0 a(d\theta) y(t+\theta) + f(y(t)) \right) dt + b dw(t) \\ & y(0) = x_0 \in R^d, \quad y(\theta) = x_1(\theta), \theta \in [-r, 0] \end{aligned} \tag{1.38}$$

Let f be a Lipschitz continuous function. It's known (see [44]) that if $H = R^d \times L^2([-r,0],R^d)$ then the H-valued process x,

$$x(t) = (y(t), y_t(\cdot)), \quad t \geq 0,$$

where $y_t(\theta) := y(t+\theta)$, $t \geq 0$, $\theta \in [-r,0]$, is a mild solution of the equation

$$dx(t) = (Ax(t) + F(x(t)))dt + Bdw(t). \tag{1.39}$$

Here the operators A, F and B are defined as follows

$$A(\phi(0),\phi) = \left(\int_{-r}^{0} a(d\theta)\phi(\theta), d\phi/d\theta\right), \tag{1.40}$$

with

$$D(A) = (\phi(0),\phi) : \phi \in W^{1,2}([-r,0],R^n),$$

and

$$F(X_0,x_1) = (f(x_0),0), (x_0,x_1) \in H, Bu = (bu,0), u \in R^m.$$

The process x will be called an extended or a complete solution of equation (1.38) and denoted by

$$x(t,x) = (y(t;x_0,x_1), y_t(.;x_0,x_1), \quad x = (x_0,x_1) \in H.$$

The authors consider the linear case of equation (1.39)

$$dx(t) = Ax(t)dt + Bdw(t), \quad x(0) = x \in H$$

with the following restriction on the discrete delay case

$$dy(t) = (a_0 y(t) + \sum_{i=1}^{N} a_i y(t+\theta_i))dt + bdw(t), \quad y(0) = x_0, y(\theta) = x_1(\theta), \theta \in [-r,0],$$

where $-r = \theta_1 < \theta_2 < ... < \theta_N < 0 = \theta_{N+1}$. The measure $a(\cdot)$ is then of the form

$$a(D) = a_0\delta_0(D) + \sum_{i=1}^{N} a_i\delta_{\theta_i}(D), \quad D \in \mathcal{B}([-r,0]).$$

Also the authors study a nonlinear case in the above form but with the additional nonlinear term $f(y(t))dt$.

The authors give an answer to the following crucial question: under what conditions does there exist a positive number ω such that the operator $A + \omega I$, A as in (1.40), is dissipative? The dissipative property is needed for the stability.

1.10. In the research report [106] (Ivanov, Swishchuk, 1999) several dynamical properties of solutions of special form nonlinear delay differential equations subject to both random initial conditions and white noise disturbances are studied. The equations are of the form

$$\nu dx(t) = [-x(t) + f(x(t-1))]dt + \sigma(x(t-1))dw(t), \tag{1.41}$$

where $x(t) \in R^n$, $\phi(t)$ is a continuous random process in R^n, σ is a bounded matrix-valued function, f is a continuous nonlinear map, and $\nu > 0$.

The invariance and global stability properties for the expectation values and the dependence of solutions on initial data and the closeness of solutions of equation (1.41) are studied. A suggested approach allows to study the global stability phenomenon as a singularly perturbed problem for $\nu > 0$.

1.11. The following paper of this review is [102] (Mohammed, 1996).

For any path $x : [-r, \infty) \to R^d$ at each $t \geq 0$ define the segment $x_t : [-r, 0] \to R^d$ by

$$x_t(s) := x(t+s) \quad a.s., \ t \geq 0, \ s \in J := [-r, 0].$$

Consider the following class of *stochastic functional differential equations* (sfde's)

$$\begin{cases} dx(t) = h(t, x_t)dt + g(t, x_t)dW(t), \quad t \geq 0 \\ x_0 = \theta \end{cases} \tag{1.42}$$

on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying *the usual conditions*; i.e. the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous and each $\mathcal{F}_t, \ t \geq 0$, contains all P-null sets in $\mathcal{F}$. Denote by $C := C([-r, 0], R^d)$ the Banach space of all continuous paths $\eta : [-r, 0] \to R^d$ given the supremum norm

$$||\eta||_C := \sup_{s \in [-r, 0]} |\eta(s)|, \quad \eta \in C.$$

In the sfde (1.42), $W(t)$ represents m-dimensional Brownian motion and $L^2(\Omega, C)$ is the Banach space of all (equivalence classes of) $(\mathcal{F}, BorelC)$-measurable maps $\theta : \Omega \to C$ which are L^2 in the Bochner sense. Give $L^2(\Omega, C)$ the Banach norm

$$||\theta||_{L^2(\Omega, C)} := \left[\int_\Omega ||\theta(\omega)||_C^2 dP(\omega) \right]^{1/2}.$$

The sfde (1.42) has a *drift coefficient* function $h : [0, T] \times L^2(\Omega, C) \to L^2(\Omega, R^d)$ and a *diffusion coefficient* function $g : [0, T] \times L^2(\Omega, C) \to L^2(\Omega, R^{d \times m})$ satisfying Hypotheses (H1) below. The *initial path* is an $\mathcal{F}_0$-measurable process $\theta \in L^2(\Omega, C; \mathcal{F}_0)$.

A *solution* of (1.42) is a measurable, sample-continuous process $x : [-r, T] \times \Omega \to R^d$ such that $x|_{[0,T]}$ is $(\mathcal{F}_t)_{0 \leq t \leq T}$-adapted, $x(s)$ is $\mathcal{F}_0$-measurable for all $s \in [-r, 0]$, and x satisfies (1.42) almost surely.

Hypotheses (H1) *(i) The coefficient functionals h and g are jointly continuous and uniformly Lipschitz in the second variable with respect to the first, i.e.*

$$||h(t, \psi_1) - h(t, \psi_2)||_{L^2(\Omega, R^d)} + ||g(t, \psi_1) - g(t, \psi_2)||_{L^2(\Omega, R^{d \times m})} \leq L||\psi_1 - \psi_2||_{L^2(\Omega, C)}$$

for all $t \in [0, T]$ and $\psi_1, \psi_2 \in L^2(\Omega, C)$. The Lipschitz constant L is independent of $t \in [0, T]$.

(ii) For each $(\mathcal{F}_t)_{0 \leq t \leq T}$-adapted process $y : [0, T] \to L^2(\Omega, C)$, the processes $h(\cdot, y(\cdot))$ and $g(\cdot, y(\cdot))$ are also $(\mathcal{F}_t)_{0 \leq t \leq T}$-adapted.

Theorem 9 *(Existence and Uniqueness) Suppose h and g satisfy Hypotheses (H1). Let $\theta \in L^2(\Omega, C; \mathcal{F}_0)$. Then the sfde (1.42) has a unique solution $x^\theta : [-r, \infty) \times \Omega \to R^d$ starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ with $[0,T] \ni t \to x_t^\theta \in C$ sample-continuous, and $x^\theta \in L^2(\Omega, C([-r,T], R^d))$ for all $T > 0$. For a given θ, uniqueness holds up to equivalence among all $(\mathcal{F}_t)_{0\le t\le T}$-adapted processes in $L^2(\Omega, C([-r,T], R^d))$.*

Note, that sfde

$$\begin{cases} dx(t) = \left(\int_{[-r,0]} x_t(s)dW(s)\right) dW(t), \quad t > 0 \\ (x(0), x_0) = (v, \eta) \in R \times L^2([-r,0], R), \end{cases} \tag{1.43}$$

does not satisfy the hypotheses underlying the classical results of Dolens-Dade, Metiver and Pelaumai, Protter, Lipster and Shiryayev, because the coefficient functional

$$\eta \to \int_{-r}^{0} \eta(s)dW(s)$$

on the right-hand side of (1.43) does not admit almost surely Lipschitz (or even linear) versions $C \to R$, *whereas* the case of sfde (1.43) is covered by the Theorem 9.

When the coefficients h and g in (1.42) factor through (deterministic) functionals

$$H : [0,T] \times C \to R^d, \quad G : [0,T] \times C \to R^{d\times m}$$

we can impose the following local Lipschitz and global linear growth conditions on the sfde

$$\begin{cases} dx(t) = H(t, x_t)dt + G(t, x_t)dW(t), \quad t \ge 0 \\ x_0 = \theta \end{cases} \tag{1.44}$$

where W is m-dimensional Brownian motion:

Hypotheses (H2) *(i) Suppose that H and G are Lipschitz on bounded sets in C uniformly in the second variable; i.e. for each integer $n \ge 1$, there exists a constant $L_n > 0$ (independent of $t \in [0,T]$) such that*

$$|H(t,\eta_1) - H(t,\eta_2)| + ||G(t,\eta_1) - G(t,\eta_2)|| \le L_n ||\eta_1 - \eta_2||_C$$

for all $t \in [0,T]$ and $\eta_1, \eta_2 \in C$ with $||\eta_1||_C \le n$, $||\eta_2||_C \le n$.

(ii) There is a constant $K > 0$ such that

$$|H(t,\eta)| + ||G(t,\eta)|| \le K(1 + ||\eta||_C)$$

for all $t \in [0,T]$ and $\eta \in C$.

Assuming Hypotheses (H2) the analogue of Theorem 9 for sfde's (1.44) holds.

The proof of the above theorem uses a classical successive approximation technique. The main steps are as follows:

(1) Truncate the coefficients of (1.44) outside any open ball of radius N in C, using globally Lipschitz partitions of unity. This, together with Step 3 below, reduces the problem of existence of a solution to the case with globally Lipschitz coefficients.

(2) Assuming globally Lipschitz coefficients, use successive approximations to obtain a unique pathwise solution of the sfde (1.42) (for the details of this argument see [102]).

(3) Under a global Lipschitz hypothesis, it is possible to show that for sfde's of type (1.44), if the coefficients agree on an open set U in C, then the trajectories starting from an initial path U must leave U at the same time and agree until they leave U. Now, truncate each coefficient in (1.44) outside an open ball of radius N and center 0 in C. Using the above local uniqueness result, one can "patch up" solutions of the truncated sfde's as N increases to infinity.

1.12. In the paper [117] (Mao, Matasov, Piunovskiy, 2000) the authors consider a stochastic delay differential equation with Markovian switching of the form

$$\begin{cases} dx(t) = f(x(t), x(t-\tau), t, r(t))dt + g(x(t), x(t-\tau), t, r(t))dW(t), \quad t \geq 0 \\ x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n), \end{cases} \tag{1.45}$$

where

$$f : R^n \times R^n \times R_+ \times S \to R^n, \quad g : R^n \times R^n \times R_+ \times S \to R^{n\times m},$$
$$S = \{1, 2, ..., N\},$$

$r(t)$ is a right-continuous Markov chain on the probability space taking values on S with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta), & if\ i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta), & if\ i = j, \end{cases}$$

where $\delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$ and that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of R_+.

The authors impose a hypothesis:

Hypotheses (H3) *Both f and g satisfy the local Lipschitz condition and the linear growth condition. That is, for each $k = 1, 2, ...,$ there is an $h_k > 0$ such that*

$$|f(x_1, y_1, t, i) - f(x_2, y_2, t, i)| + |g(x_1, y_1, t, i) - g(x_2, y_2, t, i)| \leq h_k(|x_1 - x_2| + |y_1 - y_2|)$$

for all $t \geq 0$, $i \in S$ and those $x_1, y_1, x_2, y_2 \in R^n$ with $\max\{|x_1|, |x_2|, |y_1|, |y_2|\} \leq k$*; and there is, moreover, an $h > 0$ such that*

$$|f(x, y, t, i)| + |g(x, y, t, i)| \leq h(1 + |x| + |y|)$$

for all $x, y \in R^n$, $t \geq 0$ and $i \in S$.

Theorem 10 *(Existence and Uniqueness) Under hypothesis (H3), equation* (1.45) *has a unique continuous solution* $x(t)$ *on* $t \geq -\tau$. *Moreover, for every* $p > 0$,

$$E\left[\sup_{-\tau \leq s \leq t} |x(s)|^p\right] < \infty, \quad on\ t \geq 0.$$

In the first part of the proof we use the fact that there is a sequence $\{\tau_k\}_{k\geq 0}$ of stopping times such that $0 = \tau_0 < \tau_1 < ... < \tau_k \to \infty$ and $r(t)$ is constant on every interval $[\tau_k, \tau_{k+1})$. Then, successively using an existence and uniqueness theorem for equation (1.45) on each time interval $[\tau_k \wedge T, \tau_{k+1} \wedge T]$ with initial data obtained from the previous iterations, we obtain a unique solution $x(t)$ of (1.45) at $[-\tau, T]$.

The moment estimation is obtained using the fact that the initial data is bounded.

1.13. Markov property of solutions of SDDE is considered in [102] (Mohammed, 1996). There, it was shown that the trajectory field x_t corresponds to a C-valued Markov process. The following hypotheses are needed.

Hypotheses (H4) *(i) For each* $t \geq 0$, $\mathcal{F}_t$ *is the completion of the* σ*-algebra* $\sigma\{W(u) : 0 \leq u \leq t\}$.

(ii) H *and* G *are jointly continuous and globally Lipschitz in the second variable uniformly with respect to the first:*

$$|H(t,\eta_1) - H(t,\eta_2)| + ||G(t,\eta_1) - G(t,\eta_2)|| \leq L||\eta_1 - \eta_2||_C$$

for all $t \in [0,T]$ *and* $\eta_1, \eta_2 \in C$.

Consider the sfde

$$x^{\theta,t_1}(t) = \begin{cases} \theta(0) + \int_{t_1}^t H(u, x_u^{\theta,t_1})du + \int_{t_1}^t G(u, x_u^{\theta,t_1})dW(u), & t > t_1 \\ \theta(t - t_1), & t_1 - r \leq t \leq t_1. \end{cases}$$

Let x^{θ,t_1} be its solution, starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_{t_1})$ at $t = t_1$. This gives a two-parameter family of mappings

$$T_{t_2}^{t_1} : L^2(\Omega, C; \mathcal{F}_{t_1}) \to L^2(\Omega, C; \mathcal{F}_{t_2}),\ t_1 \leq t_2$$
$$T_{t_2}^{t_1} := x_{t_2}^{\theta,t_1}, \quad \theta \in L^2(\Omega, C; \mathcal{F}_{t_1}).$$

The uniqueness of solutions to the above sfde gives the two-parameter semigroup property:

$$T_{t_2}^{t_1} \cdot T_{t_1}^0 = T_{t_2}^0, \quad t_1 \leq t_2$$

Theorem 11 *(The Markov Property) In* (1.44)*, suppose Hypotheses (H4) hold. Then the trajectory field* $\{x_t^\eta : t \geq 0, \eta \in C\}$ *is a* C*-valued Feller process with transition probabilities*

$$p(t_1, \eta, t_2, B) := P(x_{t_2}^{\eta,t_1} \in B), \quad t_1 \leq t_2, \quad B \in Borel\ C, \quad \eta \in C$$

i.e.

$$P(x_{t_2} \in B|\mathcal{F}_{t_1}) = p(t_1, x_{t_1}(\cdot), t_2, B) = P(x_{t_2} \in B|x_{t_1}) \quad a.s. \tag{1.46}$$

Furthermore, if H *and* G *do not depend on* t*, then the trajectory is time-homogeneous:*

$$p(t_1, \eta, t_2, \cdot) = p(0, \eta, t_2 - t_1, \cdot), \quad 0 \leq t_1 \leq t_2, \quad \eta \in C.$$

Proof: The first equality in (1.46) is equivalent to

$$\int_A 1_B(T_{t_2}^0(\theta)(\omega))dP(\omega) = \int_A \int_\Omega 1_B\{[T_{t_2}^{t_1}(T_{t_1}^0(\theta)(\omega'))](\omega)\}dP(\omega)dP(\omega') \tag{1.47}$$

for all $A \in \mathcal{F}_{t_1}$ and all Borel subsets B of C. In order to prove the equality (1.47), observe first that it holds when 1_B is replaced by an arbitrary uniformly continuous and bounded function $\varphi : C \to R$; that is

$$\int_A \varphi(T_{t_2}^0(\theta)(\omega))dP(\omega) = \int_A \int_\Omega \varphi\{[T_{t_2}^{t_1}(T_{t_1}^0(\theta)(\omega'))](\omega)\}dP(\omega)dP(\omega')$$

The above relation follows by approximating $T_{t_1}^0(\theta)$ using simple functions. Since C is separable and admits uniformly continuous partitions of unity, it is easy to see that (1.47) holds for all open sets B in C. By uniqueness of measure-theoretic extensions, (1.47) also holds for all Borel sets B in C.

The proof of the time-homogeneity statement is straightforward.

1.14. The notion of regularity of SDDE is considered in [95] (Mohammed, Scheutzow, 1997).

Let $M_2 := R \times L^2([-r, 0], R)$ denote the Delfour-Mitter Hilbert space with the norm

$$||(v, \eta)||_{M_2} := \left[v^2 + \int_{-r}^0 \eta(s)^2 ds\right]^{1/2}, \quad v \in R,\ \eta \in L^2([-r, 0], R).$$

Define the *trajectory* of (1.42) by $(x(t), x_t)$, $t \geq 0$, where x_t denotes the segment $x_t(s) := x(t + s)$, $s \in [-r, 0]$, $t \geq 0$. Then a liner sfde is said to be *regular* with respect to M_2 if its trajectory random field $\{(x(t), x_t) : (x(0), x_0) = (v, \eta) \in M_2,\ t \geq 0\}$ admits a (*Borel* $R^+ \times$ *Borel* $M_2 \times \mathcal{F}$; *Borel* M_2)-measurable version $X : R^+ \times M_2 \times \Omega \to M_2$ with a.a. sample functions continuous on $R^+ \times M_2$. The sfde is said to be *singular* otherwise.

Consider the one-dimensional stochastic linear delay equation

$$\begin{aligned} dx(t) &= \sigma x(t - r)dW(t), \quad t > 0, \\ (x(0), x_0) &= (v, \eta) \in M_2 := R \times L^2([-r, 0], R), \end{aligned} \tag{1.48}$$

driven by a standard Wiener process W, where $\sigma \in R$ is fixed and r is a positive delay. It is known that (1.48) is singluar with respect to M_2 for all nonzero σ [102].

Here we will examine the regularity of the more general one-dimensional linear sfde:

$$\begin{aligned} dx(t) &= \int_{[-r,0]} x(t + s)d\nu(s)dW(t), \quad t > 0 \\ (x(0), x_0) &\in M_2 := R \times L^2([-r, 0], R), \end{aligned} \tag{1.49}$$

where W is a Wiener process and ν is a fixed finte real-valued signed Borel measure on $[-r, 0]$.

Theorem 12 *(Unboundedness and Non-linearity) Let* $r > 0$*, and suppose that there exists* $\varepsilon \in (0, r)$ *such that supp* $\nu \subset [-r, -\varepsilon]$*. Suppose* $0 < t_0 \leq \varepsilon$*. For each* $k \geq 1$*, set*

$$\nu_k := \sqrt{t_0} \left| \int_{[-r,0]} \exp\left(\frac{2\pi i k s}{t_0}\right) d\nu(s) \right|.$$

Assume that

$$\sum_{k=1}^{\infty} \nu_k x^{1/\nu_k^2} = \infty \tag{1.50}$$

for all $x \in (0, 1)$*. Let* $Y : [0, \varepsilon] \times M_2 \times \Omega \to R$ *be any Borel-measurable version of the solution field* $\{x(t) : 0 \leq t \leq \varepsilon, (x(0), x_0) = (v, \eta) \in M_2\}$ *of* (1.49)*. Then for a.a.* $\omega \in \Omega$*, the map* $Y(t_0, \cdot, \omega) : M_2 \to R$ *is unbounded in every neighborhood of every point in* M_2*, and (hence) non-linear.*

Remark:
(i) Condition (1.50) of the theorem is implied by

$$\lim_{k \to \infty} \nu_k \sqrt{\log k} = \infty.$$

(ii) For the delay equation (1.48), $\nu = \delta_{-r}$, $\varepsilon = r$. In this case condition (1.50) is satisfied for every $t_0 \in (0, r]$.

Proof: The main idea is to track the solution random field of (a complexified version of) (1.49) along the classical Fourier basis:

$$\nu_k(s) = \exp\left(\frac{2\pi i k s}{t_0}\right), \quad -r \leq s \leq 0, \ k \geq 1$$

in $L^2([-r, 0], C)$. On this basis, the solution field gives an infinite family of independent Gaussian random variables. This allows us to show that no Borel measurable version of the solution field can be bounded with positive probability on an arbitrary small neighborhood of 0 in M_2, and hence on any neighborhood of any point in M_2.

Note that the pathological phenomenon in Theorem 12 is peculiar to the delay case $r > 0$. The proof of the theorem suggests that this pathology is due to the Gaussian nature of the Wiener process W coupled with the infinite-dimesionality of the state space M_2. Because of this, one may expect similar difficulties in certain types of linear stochastic differential equations driven by multidimensional white noise.

2 Stochastic stability of SDDEs

In this section we survey some results concerning the stochastic stability of stochastic differential delay equations.

2.1. We start with paper [14] (Bucy, 1965), where the stability of solutions of nonlinear difference equations containing random elements is considered. The results of the paper were announced in [11]. A generalization of these results are to be found in [15].

Guided by the Lyapunov theory for deterministic systems the author introduces the concepts of a random equilibrium point and the stability of a random solution in probability as well as an almost everywhere or almost sure asymptotic stability.

Consider the following random equation

$$x_n = f(x_{n-1}, r_{n-1}) \tag{2.1}$$

where x_0 is a given random vector variable, f is a continuous real valued vector function, and r_{n-1} is a sequence of random quantities. It's assumed that there exists almost everywhere a unique random variable x_e satisfying

$$x_e = f(x_e, r_n), \quad \forall n \geq 0$$

which is called then an equilibrium point, with function f in (2.1).

The results obtained are roughly as follows:

a) the existence of a positive definite continuous function which is a supermartingale along the solution is sufficient for the stability in probability;

b) the existence of a decreasing positive supermartingale is sufficient for the asymptotic stability almost everywhere.

By the Massera theorem [1], the existence of a Lyapunov function is a necessary condition for the appropriate type of stability.

In [3] a rather similar definition of the stability in probability is given, but the results are weaker, and the almost sure stability is not considered.

2.2. In paper [15] (Kushner, 1965) the author studies the sample asymptotic behavior and almost sure stability of the equlibrium solution of SDEs.

The author obtains a sufficient condition for the almost sure stability in terms of exponentially decaying second moments which also quarantees that the sample paths themselves decay exponencially w.p.1. It implies almost sure stability of the equilibrium solution for the inhomogeneous SDE

$$dx(t) = m(t, x(t))dt + \sigma(t, x(t))dw(t), \quad x(t) \in R^n, w(t) \in R^n$$

under the uniform Lipschitz and the uniform growth conditions on functions m and σ and the assumption

$$m(t, 0) = 0, \quad \sigma(t, 0) = 0,$$

where $x_1 = x_2 = ... = 0$ is the equilibrium solution.

Related to [15] is the work of Khasminskii [6]. Kac [5] and McKean [7] study the winding of the solution paths of simple oscillators driven by a "white noise"around the origin in the phase space of the solutions. Results similar to those obtained in [15] can also be found in [23].

In [18] the following equation is studied

$$\dot{x}(t) = Ax(t) + Bx(t + \eta(t)), \tag{2.2}$$

where A, B are matrices, $\eta(t)$ is continuous on the right, Markovian and jump process.

Sufficient conditions for the asymptotic stability in probability of solutions of equation (2.2) are given.

In [24] the author studies equation

$$\dot{x}(t) = Ax(t) + Bx(t - \eta(t)), \tag{2.3}$$

where A, B are matricies, $\eta(t)$ is a homogeneous pure jump Markov process. The asymptotic stability in probability of the zero solution of equation (2.3) is studied.

In [32] the following system of equations is investigated

$$\dot{x}(t) = y(t) \tag{2.4}$$

$$dy(t) = [-\int_0^{+\infty} y(t-s)dK_0(s) - \int_0^{+\infty} x(t-s)dK_1(s)]dt + b(t, y(t+\tau))d\xi(t),$$

$$x(\tau) = \phi(\tau), \dot{x}(\tau) = \dot{\phi}(\tau), \quad \tau \leq 0,$$

where $\xi(t)$ is a Wiener process, $K_0(s)$ and $K_1(s)$ are some measures of bounded variation on $[0, +\infty)$.

The authors prove that the trivial solution of equation (2.4) ($b(t, \phi(\tau)) = 0$, if $\phi(\tau) = 0$) is asymptotically stable in mean square.

Let

$$r(t) := E(x^2(t) + y^2(t))$$
$$\gamma(\phi) := \sup_{\tau \leq 0} E(\phi^2(\tau) + \dot{\phi}^2(\tau)).$$

Definition The trivial solution of system (2.4) :

a) is stable in mean square if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $r(t) < \epsilon$, $t \geq 0$, if and only if $\gamma(\phi) < \delta(\epsilon)$ and $\sup_{\tau \leq 0} E(\phi^4(\tau) + \dot{\phi}^4(\tau)) < +\infty$;

b) is asymptotically stable if it is stable and $\lim_{t \to +\infty} r(t) = 0$.

Theorem 13 *Suppose*

$$|b(t, \phi_1) - b(t, \phi_2|^2 \leq \int_0^{\infty} |\phi_1(-s) - \phi_2(-s)|^2 dK_2(s),$$

where K_2 *is a measure of bounded variation,* $\alpha_{ij} := \int_0^\infty s^i |dK_j(s)|$, $\beta_{ij} := \int_0^\infty s^i dK_j(s)$. *Assume also that* $K_0(s)$ *has a jump at zero of the magnitude* $a > 0$ *and* $\beta_{01} > 0, a > \int_0^{+\infty} |dK_0(s)| + \alpha_{11} + \alpha_{02}$ *and* $\alpha_{01} + \alpha_{21} + \alpha_{12} < +\infty$. *Then the trivial solution of equation (2.4) is asymptotic stable in mean square.*

The proof uses a Lyapunov function of the following form

$$V(x(t+\tau), y(t+\tau)) = y(t) + \int_0^\infty x(t-s)dK_0(s) - \int_0^\infty dK_1(s) \int_{t-s}^t x(t_1)dt_1.$$

In [38], for the equation

$$\dot{x}(t) = f(t, x_t(\theta))$$

with retardation $x_t(\theta) = x(t+\theta)$, $0 \geq \theta \geq \eta(t,\omega)$, $-r \leq \eta \leq 0$, η is a stochastic process, the stability of solutions in mean is considered.

In paper [29] an equation of the type (2.4) is considered and the asymptotic stability of the trivial solution in mean square is studied.

In [25], for the equation

$$\dot{x}(t) = F(x(t+\theta), \xi(t,\omega)),$$

where $-\infty < \theta \leq 0$ and $\xi(t)$ is stationary process, the existence of stationary and periodic solutions is studied.

2.3. Stability of solutions of equation (1.21) is studied in [27] (Kushner, 1968). Various definitions concerning the stochastic stability are introduced in [15, 26]. The following results are extensions of those in [26], where the state space is a Euclidian space.

Theorem 14 *Let x_t be a continuous on the right strong Markov process defined on the topological state space $(C, \mathcal{C}, \mathcal{B})$ with weak infinitesimal operator $\tilde{A}$. Let the norm $||\cdot||$ generates $\mathcal{C}$, and let the nonnegative continuous real-valued function $V(x) \in \mathcal{D}(\tilde{A})$. Let $Q = \{x : V(x) < q\}$ and let $\tau := \inf\{t : x_t \notin Q\}$. Set $\tau = \infty$ if $x_t \in Q$ for all $t < +\infty$. Let $\tilde{A}V(x) = -k(x) < 0$ in Q. Then for $x = x_0 \in Q$*

(B1) $V(x_{t\wedge\tau}) = \omega_t$ is a nonnegative supermartingale;

(B2) $P_x \sup_{+\infty > t > 0} V(x_t) \leq q \leq V(x)/q$;

(B3) $V(x_{t\wedge\tau}) \to v \geq 0$ w.p.1;

If, in addition,

i) K is uniformly continuous on the nonempty open set $R_{\hat{\delta}} = x : k(x) < \hat{\delta} \cap Q$ for some $\hat{\delta} > 0$, and

ii) for all sufficiently large but finite Markov times t and all sufficiently small ϵ one has

$$P_x\{\max_{t+h \geq s \geq t} ||x_s - x_t|| \geq \epsilon, x_r \in Q, r \leq t\} \to 0$$

as $h \to 0$ uniformly in t for sufficiently large t and any $x \in Q$,

then

(B4) $k(x_t) \to 0$ w.p.1 (relative to $\Omega_Q = \{\omega : \sup_{\infty > t \geq 0} V(x_t) < q\}$).

Theorem 15 *Assume A1), A2) and A4) of subsection 1.1. Let $V(x)$ be a continuous nonnegative real valued function on C. Suppose that* *(iii) there is a bounded open set B such that*

$$x_0 = x \in Q = \{x : V(x) < q\} \cap B,$$

and $\sup_{t>s\geq 0} V(x_s) < q$ *imply that* $x_s \in Q$ *for all* $0 < s < t$.

Let $V(x) \in \mathcal{D}(\tilde{A}_Q)$ *and* $\tilde{A}_Q V(x) = -k(x) \leq 0$ *in* Q.

Then B1)-B3) hold, and $P(\Omega_Q) \geq 1 - V(x)/q$.

If k *is uniformly continuous on* $R_{\hat{\delta}} = x : k(x) < \hat{\delta}$ *for some* $\hat{\delta} > 0$ *then* $k(x_t) \to 0$ *w.p.1 (relative to* Ω_Q*).*

2.4. In book [51] (Khasminskii, 1980) the author deals with some stability results of stochastic systems which appeared since the Russian edition of the book [30] was published.

For the differential equation

$$\dot{y}(t) = F(x(t))y(t),$$

where $y(t) \in R^n$, $F(x)$ is an $n \times n$ matrix, $x(t)$ is a Markov process with infinitesimal operator L, the author gives a review of the moment stability and the almost sure stability for linear systems of equations whose coefficients are Markov processes.

Also, the author studies the almost sure stability of paths of the one-dimensional diffusion process

$$dx(t) = b(x((t))dt + \sigma(x(t))dw(t). \tag{2.5}$$

The basic result is this one. If $x(t)$ in (2.5) is a recurrent process in R^n, then the process $x(t)$ is an almost sure stable in the large.

The author also considers a reduction principle for a two-component diffusion process.

2.5. In the survey paper [82] (Mandrekar, 1994) the author presents a review of the results related to some extensions of Lyapunov methods in the stability theory for the solutions of infinite-dimensional (stochastic and deterministic) evolution equations. As a consequence he can derive results on the stability of solutions of SDDEs of the type (1.37) by using the spectral theory of operators generated by the equation.

The same methods are used in [92] for SDDEs of the types (1.38) and (1.39).

2.6. We continue this section by reviewing of [117] (Mao, Matasov, Piunovskiy, 2000). Consider a stochastic delay differential equation with Markovian switching (1.45):

$$\begin{cases} dx(t) = f(x(t), x(t-\tau), t, r(t))dt + g(x(t), x(t-\tau), t, r(t))dW(t), & t \geq 0 \\ x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n), \end{cases} \tag{2.6}$$

Denote the solution of this equation by $x(t, \xi)$. For the purpose of stability we may asssume, without loss of generality, that $f(0, 0, t, i) = 0$ and $g(0, 0, t, i) = 0$ for all $t \geq 0$ and $i \in S$. Therefore, in this case, equation (2.6) admits a trivial solution $x(t, 0) = 0$.

Let $C^{2,1}(R^n \times R_+ \times S; R_+)$ denote a family of all non-negative functions V on $R^n \times R_+ \times S$ which are twice continuously differentiable in x and once differentiable in t. For any

$V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ define $\mathcal{L}V : R^n \times R^n \times R_+ \times S \to R$ by

$$\begin{aligned}\mathcal{L}V(x,y,t,i) =& V_t(x,t,i) + V_x(x,t,i)\ f(x,y,t,i)+\\ &+ \frac{1}{2}trace\{g^T(x,y,t,i)V_{xx}(x,t,i)g(x,y,t,i)\} + \sum_{j=1}^{N} \gamma_{ij}V(x,t,j).\end{aligned} \tag{2.7}$$

Theorem 16 *(p-th Moment Exponentional Stability) Let the hypothesis (H3) hold. Let* p, c_1, c_2 *be positive numbers and* $\lambda_1 > \lambda_2 \geq 0$. *Assume that there exists a function* $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ *such that*

$$c_1|x|^p \leq V(x,t,i) \leq c_2|x|^p$$

for all $(x,t,i) \in R^n \times R_+ \times S$, *and*

$$\mathcal{L}V(x,y,t,i) \leq -\lambda_1|x|^p + \lambda_2|y|^p$$

for all $(x,y,t,i) \in R^n \times R^n \times R_+ \times S$. *Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log(E|x(t,\xi)|^p) \leq -\gamma$$

for all $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$, *where* $\gamma > 0$ *is the unique root of the equation*

$$\gamma(c_2 + \tau\lambda_2 e^{\gamma\tau}) = \lambda_1 - \lambda_2$$

In other words, the trivial solution of (2.6) *is p-th moment exponentially stable with Lyapunov exponent not greater than* $-\gamma$.

Proof: Define $U(x,t,i) \in C^{2,1}(R^n \times R_+ \times S; R_+)$ by

$$U(x,t,i) = e^{\gamma t}\left[V(x,t,i) + \lambda_2 \int_{t-\tau}^{t} E|x(s)|^p ds\right].$$

Then using Ito's formula

$$EU(x(t),t,r(t)) = EU(x(0),0,r(0)) + E\int_0^t \mathcal{L}U(x(s),x(s-\tau),s,r(s))ds$$

obtain the following estimate

$$EU(x(t),t,r(t)) \leq (c_2 + \tau\lambda_2(1+e^{\gamma\tau}))E||\xi||^p.$$

Also, from the assumptions we have another estimate

$$EU(x(t),t,r(t)) \geq e^{\gamma t}E|x(t)|^p.$$

Consequently

$$E|x(t)|^p \leq \frac{1}{c_1}(c_2 + \tau\lambda_2(1+e^{\gamma\tau}))e^{-\gamma\tau},$$

and the required assertion follows.

Theorem 17 *(Almost sure exponentional stability) Let the hypothesis (H3) hold. Assume that there is a constant* $K > 0$ *such that for all* $(x, y, t, i) \in R^n \times R^n \times R_+ \times S$,

$$|f(x, y, t, i)| \vee |g(x, y, t, i)| \leq K(|x| + |y|).$$

Let $p > 0$. *Assume that the trivial solution of* (2.6) *is p-th moment exponentially stable, that is, there is a positive constant* γ *such that*

$$\limsup_{t\to\infty} \frac{1}{t} \log(E|x(t, \xi)|^p) \leq -\gamma$$

for all $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$. *Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t, \xi)|) \leq -\frac{\gamma}{p} \ a.s.$$

i.e., p-th moment exponential stability implies almost sure exponential stability.

Proof: Fix the initial value ξ arbitrarily and write $x(t, \xi) = x(t)$. Let $\varepsilon \in (0, \gamma/2)$ be arbitrary. By the p-the moment exponential estimate and the theorem 10, there is a positive constant k such that

$$E|x(t)|^p \leq ke^{-(\gamma-\varepsilon)t}, \quad for\ t \geq -\tau.$$

Here we skip an estimation routine of $E[\sup_{(k-1)\sigma\leq t\leq k\sigma} |x(t)|^p]$. Finally, the following inequality is obtained:

$$P\left\{\omega : \sup_{(k-1)\sigma\leq t\leq k\sigma} |x(t)| > e^{-(\gamma-2\varepsilon)k\sigma/p}\right\} \leq C(k + 1)e^{-\varepsilon k\sigma}.$$

for some $C > 0$. In view of the well-known Borel-Cantelli lemma, we see that for almost all $\omega \in \Omega$,

$$\sup_{(k-1)\sigma\leq t\leq k\sigma} |x(t)| \leq e^{-(\gamma-2\varepsilon)k\sigma/p}$$

holds for all but finitely many k. Hence, there exists a $k_0(\omega)$, for all $\omega \in \Omega$ excluding a P-null set, for which the above inequality holds whenever $k \geq k_0$. Consequently, for almost all $\omega \in \Omega$,

$$\frac{1}{t} \log(|x(t)|) \leq -\frac{(\gamma - 2\varepsilon)k\sigma}{pt} \leq -\frac{\gamma - 2\varepsilon}{p}$$

if $(k - 1)\sigma \leq t \leq k\sigma$. Therefore,

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t)|) \leq -\frac{\gamma - 2\varepsilon}{p} \ a.s.$$

and the required inequality follows by letting $\varepsilon \to 0$.

2.7. Now we focus on the article [116] (Mao, 2000). The subject of this paper is stability of SDDE with Markovian switching of the following form

$$dx(t) = f(x(t), x(t - \delta(t)), t, r(t))dt + g(x(t), x(t - \delta(t)), t, r(t))dW(t) \tag{2.8}$$

on $t \geq 0$ with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0], R^n)$, where

$$f : R^n \times R^n \times R_+ \times S \to R^n \quad and \quad g : R^n \times R^n \times R_+ \times S \to R^{n\times m}.$$

We impose the following hypotheses:

Hypotheses (H5) *(i) Both f and g satisfy the local Lipschitz condition and the linear growth condition. That is, for each $k = 1, 2, ...,$ there is an $h_k > 0$ such that*

$$|f(x_1, y_1, t, i) - f(x_2, y_2, t, i)| + |g(x_1, y_1, t, i) - g(x_2, y_2, t, i)| \leq h_k(|x_1 - x_2| + |y_1 - y_2|)$$

for all $t \geq 0$, $i \in S$ and those $x_1, y_1, x_2, y_2 \in R^n$ with $\max\{|x_1|, |x_2|, |y_1|, |y_2|\} \leq k$.

(ii) For every $i \in S$, there are constants $\alpha_i \in R$ and $\rho_i, \sigma_i, \beta_i \geq 0$ such that

$$2|x^T f(x, 0, t, i)| \leq \alpha_i|x|^2, \quad |f(x, 0, t, i) - f(x, y, t, i)| \leq \rho_i|y|,$$
$$|g(x, y, t, i)|^2 \leq \sigma_i|x|^2 + \beta|y|^2$$

for all $x, y \in R^n$ and $t > 0$.

Under these hypotheses, for any given initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0], R^n)$ equation (2.8) has a unique continuous solution denoted by $x(t, \xi)$ on $t \geq -\tau$. Let us define

$$\mathcal{A} = diag(-(\alpha_1 + \rho_1 + \sigma_1), ..., -(\alpha_N + \rho_N + \sigma_N)) - \Gamma$$

where Γ is a generator of Markov chain $r(t)$. We shall assume that the matrix $\mathcal{A} = (a_{ij})$ has all positive leading principal minors, that is,

$$\det (\ a_{ij} \mid 1 \leq i, j \leq k\ ) > 0 \quad for\ all\ 1 \leq k \leq N.$$

Also, we assume

$$(1/(\rho_1 + \beta_1), \ldots, 1/(\rho_N + \beta_N))^T \ > \ \mathcal{A}^{-1}\vec{1}.$$

We further define

$$(b_1, \ldots, b_N)^T = \mathcal{A}^{-1}\vec{1}, \quad k = \max_{1\leq i\leq N} b_i(\rho_i + \beta_i), \quad \theta = \frac{1-k}{2k}, \quad \lambda = \frac{1+k}{2}$$
$$q_i = (1 + \theta)b_i, \quad \hat{q} = \min_{1\leq i\leq N} q_i, \quad \check{q} = \max_{1\leq i\leq N} q_i.$$

With these notations we can now state the sufficient criteria for the exponential stability of equation (2.8).

Theorem 18 *Let hypotheses (H5) hold. Suppose all the assumptions on matrix $\mathcal{A}$ hold too. Then the trivial solution of* (2.8) *is exponentially stable in mean square if one of the following conditions is satisfied:*

(1) $\sup_{t\in R_+} \dot{\delta}(t) < 1 - \frac{\lambda}{1+\theta}$;

(2) $\dot{\delta}(t) < 1$ for all $t \geq 0$ and

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t [0 \vee \dot{\delta}(u)]du < \frac{\gamma\hat{q}}{\lambda}e^{-\gamma\tau}$$

where $\gamma > 0$ is the unique solution to the equation $1 + \theta = \gamma\check{q} + \lambda e^{\gamma\tau}$.

(3) $\hat{q}(1 + \theta) > \lambda\check{q}\exp[\tau(1 + \theta)/\check{q}]$.

2.8. The article [118] (Mao, Shaikhet, 2000) discusses an exponential stability of the following SDDE with Markovian switching:

$$dx(t) = f(x(t), x(t-\tau_1), t, r(t))dt + g(x(t), x(t-\tau_2), t, r(t))dW(t) \tag{2.9}$$

on $t \geq 0$ with initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0], R^n)$, $\tau = \tau_1 \vee \tau_2$, where

$$f : R^n \times R^n \times R_+ \times S \to R^n \quad \textit{and} \quad g : R^n \times R^n \times R_+ \times S \to R^{n\times m}.$$

As a standing hypothesis, we assume that both f and g satisfy the local Lipschitz condition and the linear growth condition which assures the existence and uniqueness of the solution with initial data ξ.

Hypotheses (H6) *(i) For every $i \in S$, there are constants $\alpha_i \in R$ and $\beta_i, \gamma_i \geq 0$ such that*

$$2|x^T f(x, x, t, i)| \leq \alpha_i |x|^2 \quad \textit{and} \quad |g(x, z, t, i)|^2 \leq \beta_i |x|^2 + \delta_i |z|^2$$

for all $x, z \in R^n$ and $t \geq 0$.

(ii) There are nonnegative constants K_1, K_2 and K_3 such that

$$|f(x, x, t, i) - f(x, y, t, i)|^2 \leq K_1 |x - y|^2 \quad \textit{and} \quad |f(x, y, t, i)|^2 \leq K_2 |x|^2 + K_3 |y|^2$$

for all $x, y \in R^n$ and $i \in S$.

The trivial solution is said to be exponentially stable in mean square if

$$\limsup_{t\to\infty} \frac{1}{t} \log(E|x(t, \xi)|^2) < 0$$

for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0], R^n)$.

Theorem 19 *Let hypotheses (H6) hold. Set*

$$\check{q} = \max_{1\leq i\leq N} q_i, \quad \check{\beta} = \max_{1\leq i\leq N} \beta_i, \quad \check{\delta} = \max_{1\leq i\leq N} \delta_i.$$

Assume that there are positive constants $q_1, q_2, \ldots, q_N$ and θ such that $\lambda_1 > \lambda_2$, where

$$\lambda_1 = \min_{1\leq i\leq N} \left(-[\alpha_i + \beta_i + \theta] q_i - \sum_{j=1}^{N} \gamma_{ij} q_j \right), \quad \lambda_2 = \max_{1\leq i\leq N} \delta_i q_i.$$

Let

$$\tau^* = \frac{1}{2(K_2 + K_3)} \left(\sqrt{(\check{\beta} + \check{\delta})^2 + 2\theta(\lambda_1 - \lambda_2)(K_2 + K_3)/\check{q} K_1} - \check{\beta} - \check{\delta} \right).$$

If the time lag $\tau_1 < \tau^*$ though the time lag τ_2 is arbitrary, then the trivial solution of equation (2.9) *is exponentially stable in mean square.*

2.9. The article [125] (Swishchuk, Kazmerchuk, 2002) considers the following model.

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ be a complete probability space with filtration satisfying usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Define stochastic processes $\{x(t), t \in [-h, T]\} \in R^n$, $\{\phi(\theta), \theta \in [-h, 0]\} \in R^n$, scalar Brownian motion $\{W(t), t \in [0, T]\}$, centralized Poisson measure $\{\nu(dy, dt), t \in [0, T], y \in [-1, +\infty)\}$ with parameter $\Pi(dy)dt$ and for all $t \in [0, T]$:

$$\begin{aligned} x(t) =&\phi(0) + \int_0^t [a(r(s))x(s) + \mu(r(s))x(s-\tau)]ds+ \\ &+ \int_0^t \sigma(r(s))x(s-\rho)dW(s) + \int_0^t \int_{-1}^{+\infty} yx(s)\nu(dy, ds); \end{aligned} \tag{2.10}$$

and $x(t) = \phi(t)$ for $t \in [-h, 0]$. Here: $a(\cdot), \mu(\cdot), \sigma(\cdot)$ are matrix maps with dimension $n \times n$ acting from the set $S = \{1, 2, ..., N\}$; $\tau > 0$, and $\rho > 0$; $\{r(t), t \in [0, +\infty)\}$ is Markovian chain taking values at the set S with generator $\Gamma = (\gamma_{ij})_{N\times N}$:

$$P(r(t+\delta) = j | r(t) = i) = \begin{cases} \gamma_{ij}\delta + o(\delta), & i \neq j \\ 1 + \gamma_{ii}\delta + o(\delta), & i = j \end{cases}$$

where $\delta > 0, \gamma_{ij} \geq 0$ and $\gamma_{ii} = -\sum_{i\neq j} \gamma_{ij}$. Assume $r(\cdot)$ is independent of $W(\cdot)$.

Assign $C^{2,1}(R^n \times R_+ \times S; R_+)$ the family of all nonnegative functions $V(x, t, i)$ at $R^n \times R_+ \times S$ which have a contionuous second derivative by x and have a continuous derivative by t. For $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ introduce an operator $LV : R^n \times R_+ \times S \to R$ by rule:

$$\begin{aligned} LV(x, t, i) =&V_t(x, t, i) + V_x(x, t, i) \cdot [a(i)x(t) + \mu(i)x(t-\tau)] \\ &+ \frac{1}{2} tr[x^T(t-\rho)\sigma^T(i)V_{xx}(x, t, i)\sigma(i)x(t-\rho)] + \sum_{j=1}^N \gamma_{ij}V(x, t, j) \\ &+ \int_{-1}^{\infty} [V(x + yx, t, i) - V(x, t, i) - V_x(x, t, i)yx] \, \Pi(dy) \end{aligned} \tag{2.11}$$

Theorem 20 *Assume p, c_1, c_2 are positive integers and $\lambda_1 > 0$ and there a exists function $V(x, t, i) \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that:*

$$c_1\|x\|^p \leq V(x, t, i) \leq c_2\|x\|^p$$

and

$$LV(x, t, i) \leq -\lambda_1\|x\|^p$$

for all $(x, t, i) \in R^n \times R_+ \times S$. Then

$$\limsup_{t\to\infty} \frac{1}{t} \ln(E|x(t)|^p) \leq -\gamma,$$

where $x(t)$ is the solution of (2.10) *with initial data $\phi(t)$ and $\gamma > 0$ is defined by $\gamma = \lambda_1/c_2$. In other words, a trivial solution of* (2.10) *is p-exponentially stable and p-exponent of Liapunov no greater then $-\gamma$.*

3 Qualitative theory of stochastic dynamical systems (SDSs)

Qualitative theory of SDSs studies the dynamical behavior of solutions of stochastic differential equations on the entire time interval, e.g. such as recurrence and stability properties.

3.1. In the survey paper [53] (Arnold, 1981) the author considers some problems and results of the qualitative theory of SDEs and provides a reference to papers where more details can be found. There the SDSs are given by ordinary differential equations of the form $\dot{x} = f(x, \xi)$ with a random noise process ξ in the right hand side and a random initial condition $x(0) = x$. An emphasis is made on nonlinear systems (icluding multiplicative noise linear systems) and on stationary and Markovian noises. A related reference is [54].

3.2. A general framework of a stochastic version of the bifurcation theory is proposed in [78] (Arnold, Boxler, 1992). The concepts are supported by one dimensional examples which are perturbed versions of deterministic differential equations exhibiting the elementary bifurcation scenarious. As in some cases solutions may grow to infinity in a finite time local SDSs have to be introduced then.

3.3. A criterion for the existence of global random attractors for RDSs is established in [83] (Crauel, Flandoli, 1994). The existence of invariant Markov measures supported by the random attractor is proved. For stochastic partial differential equations (SPDEs) this yields invariant measures for the associated Markov semigroup. The results are applied to reaction diffusion equations with additive white noise and to Navier-Stokes equations with multiplicative and additive white noises.

3.4. Problems in the intersection of stochastic analysis and the theory of dynamical systems are discussed in paper [85] (Arnold, 1995). The first problem discussed there is:

which SDEs generate SDSs (cocycles), and conversely, which cocycles have generators that are SDEs?

By calling a process with stationary increments a helix, the answer can be given in the form

$$\text{semimartingale cocycle} = \exp\{\text{semimartingale helix}\}.$$

The second part of the paper provides a list of anticipative problems which arise if the multiplicative ergodic theorem (MET) is used. As the first application of the MET a linear SDE with a simple Lyapunov spectrum is decoupled. The second application of the MET is a derivation of Furstenberg-Khasminskii formulas including the one for the lower Lyapunov function.

3.5. A review of some almost sure and moment stability theory for linear SDEs, with an indication which of the results carry over to equations with homogeneous vector fields, is given in [86] (Arnold, Khasminskii, 1995). The carry over is valid in particular for the estimates of Baxendale for the probability with which an almost surely stable system exceeds a threshold. Those estimates are controlled by a number γ_0 which is called the stability index and which has the property that the γ_0-th power of the solution is "almost" a martingale.

The authors prove that these estimates remain true for a nonlinear equation which is close to a homogeneous one in a neighborhood of zero.

3.6. In paper [87] (Pinsky, 1995) the author shows that the Lyapunov exponent has the following property: for any Lyapunov-stable linear SDE the Lyapunov exponent is invariant under all nonlinear perturbations.

A similar result is proved for Lyapunov-unstable systems.

3.7. A number of the stochastic stability theorems for random evolutions (which are random operator dynamical systems) and for evolutionary stochastic systems (which are their realizations) in averaging and diffusion approximation schemes are proved in book [94] (Swishchuk, 1997) by using the martingale approach.

4 Numerical Approximation and Parameter Estimation

4.1. The first article of this section is [113] (Kuchler, Platen, 2000). This paper introduces an approach for the derivation of discrete time approximations for solutions of stochastic differential equations with time delay. The following SDDE is considered.

$$dX(t) = a(t, X(t), X(t-r))dt + \sum_{j=1}^{m} b^j(t, X(t), X(t-r))dW^j(t) \tag{4.1}$$

for $t \in [0, T]$ and $r > 0$. Here the given d-dimensional initial segment ξ is assumed to be right continiuous having left hand limits. Assume Lipschitz continuity condition and Growth condition of the coefficients in (4.1).

We introduce the time step size $\Delta_l = r/l$ for $l \geq 2$. Let us then define an equidistant time discretisation $\tau_{\Delta_l} = \{\tau_n : n = -l, -l+1, \dots, 0, 1, \dots, N\}$ of the time interval $[-r, T]$ with $\tau_n = n\Delta_l$.

Let us consider a process $Y^{\Delta_l} = \{Y^{\Delta_l}(t), t \in [-r, T]\}$ that is right continuous with left hand limits. We shall call Y^{Δ_l} a *discrete time approximation* with step size Δ_l if it is based on a time discretisation $(\tau)_{\Delta_l}$ such that for each $n \in \{1, \dots, N\}$, the random variable $Y^{\Delta_l}(\tau_n)$ is $\mathcal{F}_{\tau_n}$-measurable and $Y^{\Delta_l}(\tau_{n+1})$ can be expressed as a function of $Y^{\Delta_l}(\tau_{-l}), Y^{\Delta_l}(\tau_{-l+1}), \dots, Y^{\Delta_l}(\tau_n)$, discretisation time τ_n and a finite number of $\mathcal{F}_{\tau_{n+1}}$-measurable random variables $Z_{n+1,j}, j = 1, \dots, i$.

We shall say that a discrete time approximation Y^{Δ_l} *converges strongly with order* $\gamma > 0$ at the time T if there exists a positive constant C, which does not depend on Δ_l and $L \in \{2, 3, \dots\}$ such that

$$\varepsilon(\Delta_l) = E(|X_T - Y^{\Delta_l}(T)|) \leq C(\Delta_l)^\gamma$$

for each $l \geq L$. Furthermore, Y^{Δ_l} *converges strongly* towards X at the time T if

$$\lim_{l \to \infty} \varepsilon(\Delta_l) = 0.$$

Let us now introduce the Euler approximation which is defined by

$$Y_{n+1} = Y_n + a(\tau_n, Y_n, Y_{n-l})\Delta + \sum_{j=1}^{m} b^j(\tau_n, Y_n, Y_{n-1})\Delta W_n^j \tag{4.2}$$

with the Wiener process increments

$$\Delta W_n^j = W_n^j(\tau_{n+1}) - W_n^j(\tau_n)$$

and with initial values

$$Y_i = X(\tau_i).$$

Theorem 21 *Suppose $a, b^1, b^2, \dots, b^m$ are time homogeneous and that the Lipschitz condition and the Growth condition hold and that the initial segment ξ is in $L^2(\Omega, C, \mathcal{F}_0)$. Then the Euler approximation* (4.2) *converges strongly with order $\gamma = 0.5$.*

One can observe, as already known for SDE's, that higher order approximations, in general, have to involve multiple stochastic integrals of higher multiplicity. For more details, see the paper.

4.2. The next article is [112] (Kuchler, Kutoyants, 2000).

We observe the trajectory $X^T = \{X(t), 0 \leq t \leq T\}$ of the diffusion-type process $X(t)$ given by

$$dX(t) = bX(t - \vartheta)dt + dW(t), \quad X(s) = X_0(s), \quad \pi/2b \leq s \leq 0, \ t \geq 0 \tag{4.3}$$

where b is a negative constant, ϑ is an unknown parameter from the set $\Theta = [0, -(eb)^{-1}]$ (we denote $\Theta_0 = (0, -(eb)^{-1})$ as well) and $X_0(s)$ is a known continuous function. We are interested in asymptotic properties of the maximum likelihood and Bayes estimator of the parameter ϑ as $T \to \infty$.

Let us denote $P_\vartheta^{(T)}$ the measure induced in the measurable space of continuous on $[0, T]$ functions $(\mathcal{G}[0, T], \mathcal{B}[0, T])$ by the solution of the equation (4.3). Here $\mathcal{B}[0, T]$ denotes the σ-algebra of Borel subsets of $\mathcal{G}[0, T]$ with endowed supremum norm. The solutions of (4.3) corresponding to different values of $\vartheta \in \Theta$ are continuous, thus all the measures $\{T_\vartheta^{(T)}, \vartheta \in \Theta\}$ are equivalent and the likelihood ratio process

$$L(\vartheta, \vartheta_1, X^T) = \frac{dP_\vartheta^{(T)}}{dP_{\vartheta_1}^{(T)}}(X^T), \quad \vartheta \in \Theta$$

is given by the formula

$$\begin{aligned} L(\vartheta, \vartheta_1, X^T) = \exp\{b \int_0^T (X(t - \vartheta) - X(t - \vartheta_1))dX(t) - \\ \frac{b^2}{2} \int_0^T (X^2(t - \vartheta) - X^2(t - \vartheta_1)dt\}, \end{aligned} \tag{4.4}$$

where ϑ_1 is some fixed value (Lipster, Shiryayev, 1977).

The maximum likelihood estimator (MLE) $\hat{\vartheta}_T$ is defined as a solution of the equation

$$L(\hat{\vartheta}, \vartheta_1, X^T) = \sup_{\vartheta \in \Theta} L(\vartheta, \vartheta_1, X^T).$$

Below we suppose that the true value ϑ is an interior point of Θ. We will see later that the function $L(\vartheta, \vartheta_1, X^T), \vartheta \in \Theta$ is continuous with probability one. Hence, the MLE $\hat{\vartheta}_T$ exists and belongs to Θ. If (4.4) has more than one solution than we take any one as a MLE.

To introduce the Bayesian estimator we suppose that ϑ is a *random variable* with a prior density $\pi(y), y \in \Theta$ and the loss function is quadratic. Then the Bayes estimator (BE) $\tilde{\vartheta}_T$ is a posterior mean

$$\tilde{\vartheta}_T = \int_\Theta y p(y|X^T) dy,$$

where the posterior density $p(y|X^T)$ is given by

$$p(y|X^T) = \frac{\pi(y) L(y, \vartheta_1, X^T)}{\int_\Theta \pi(v) L(v, \vartheta_1, X^T) dv}$$

The limit distributions of these estimators can be expressed through the following stochastic process

$$Z(u) = \exp\left\{ b\tilde{W}(u) - \frac{1}{2}|u|b^2 \right\}, \quad u \in R,$$

where $\tilde{W}(u) = W_+(u)$ for $u \geq 0$ and $\tilde{W}(u) = W_-(-u)$ for $u < 0$ and $W_+(u)$, $W_-(u)$, $u \geq 0$ are two independent standard Wiener processes. Let us introduce two random variables:

$$\xi = \arg\sup_u Z(u), \quad \zeta = \frac{\int_{-\infty}^{\infty} y Z(y) dy}{\int_{-\infty}^{\infty} Z(u) du}$$

Let K be an arbitrary compact subset of Θ_0. The following theorems describe the asymptotic behaviour of MLE and BE estimators.

Theorem 22 *The MLE $\hat{\vartheta}_T$ of parameter $\vartheta \in \Theta_0$ is uniform and consistent in K, the normed difference $T(\hat{\vartheta}_T - \vartheta)$ converges uniformly (in $\vartheta \in K$) in distribution to the random variable ξ and for any $p > 0$ the following holds*

$$\lim_{T \to \infty} E_\vartheta |T(\hat{\vartheta}_T - \vartheta)|^p = E|\xi|^p.$$

Theorem 23 *Let the prior density $\pi(y), y \in \Theta$ be a positive, continuous function, then the BE $\tilde{\vartheta}_T$ is uniformly consistent in K, the normed difference $T(\tilde{\vartheta}_T - \vartheta)$ converges uniformly (in $\vartheta \in K$) in distribution to the random variable ζ and for any $p > 0$ the following holds*

$$\lim_{T \to \infty} E_\vartheta |T(\tilde{\vartheta}_T - \vartheta)|^p = E|\zeta|^p.$$

Moreover, the BE is asymptotically efficient.

5 Applications

5.1. The paper [104] (Shaikhet, 1998) considers a predator-prey model which is described by the following system of equations

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left(a - \int_0^\infty f_1(s)x_1(t-s)ds - \int_0^\infty f_2(s)x_2(t-s)ds\right),\\ \dot{x}_2(t) &= -bx_2(t) + \int_0^\infty g_1(s)x_1(t-s)ds \int_0^\infty g_2(s)x_2(t-s)ds,\end{aligned} \tag{5.1}$$

where x_1 and x_2 are densities of predator and prey populations, resp. Assume, a and b are positive constants, f_i and g_i, $i = 1, 2$, are non-negative functions, such that

$$\begin{aligned}\alpha_i &= \int_0^\infty f_i ds < \infty, \quad \beta_i = \int_0^\infty g_i ds < \infty,\\ p_i &= \int_0^\infty s f_i ds < \infty, \quad q_i = \int_0^\infty s g_i ds < \infty.\end{aligned}$$

It is easy to see that the positive equilibrium (x_1^*, x_2^*) of the system (5.1) is given by

$$x_1^* = \frac{b}{\beta_1\beta_2}, \quad x_2^* = \frac{a - \alpha_1 x_1^*}{\alpha_2} = \frac{a\beta_1\beta_2 - b\alpha_1}{\alpha_2\beta_1\beta_2}$$

provided that (everywhere below this condition is assumed hold)

$$a\beta_1\beta_2 > b\alpha_1.$$

Note, that some particular class of model (5.1) were often considered before, e.g.

$$f_i(s) = a_i\delta(s), \quad g_i(s) = b_i\delta(s-h), \quad i = 1, 2, \quad h \geq 0,$$

where $\delta(s)$ is the Dirac function. Then the system (5.1) has the form

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)(a - a_1x_1(t) - a_2x_2(t)),\\ \dot{x}_2(t) &= -bx_2(t) + b_1b_2x_1(t-h)x_2(t-h).\end{aligned}$$

Let us assume that the system (5.1) is exposed by stochastic perturbations which are of white noise type and are directly proportional to $y_i(t) = x_i(t) - x_i^*$. That is, the system has the following look

$$\begin{aligned}\dot{y}_1(t) =& -(y_1(t) + x_1^*)\left(\int_0^\infty f_1(s)y_1(t-s)ds + \int_0^\infty f_2(s)y_2(t-s)ds\right) + \sigma_1 y_1(t)\dot{W}_1(t),\\ \dot{y}_2(t) =& -by_2(t) + x_2^*\int_0^\infty g_1(s)y_1(t-s)ds + x_1^*\int_0^\infty g_2(s)y_2(t-s)ds +\\ &+ \int_0^\infty g_1(s)y_1(t-s)ds\int_0^\infty g_2(s)y_2(t-s)ds + \sigma_2 y_2(t)\dot{W}_2(t).\end{aligned} \tag{5.2}$$

The stochastic stability of the trivial solution of (5.2) in the probability sense was established by constructing appropriate Lyapunov functionals. The following theorem states the result.

Theorem 24 *Let* $\beta_1 > 1$ *and there are positive constants* γ_1, γ_2, γ_3, γ_4 *such that*

$$2\alpha_1 x_1^*(1 - p_1 x_1^*) > \sigma_1^2 + (\gamma_1 + \gamma_3 p_1 x_1^*)\alpha_2 x_1^* + (\gamma_2 + \gamma_4 q_2 x_1^*)\beta_1 x_2^*,$$
$$2(b - \beta_2 x_1^*)(1 - q_2 x_1^*) > \sigma_2^2 + (\gamma_1^{-1} + \gamma_3^{-1} p_1 x_1^*)\alpha_2 x_1^* + (\gamma_2^{-1} + \gamma_4^{-1} q_2 x_1^*)\beta_1 x_2^*.$$

Then the zero solution of the system (5.2) *is stable in the probability sense.*

5.2. In the article [125] (Swishchuk, Kazmerchuk, 2002) the following equation is assumed to determine a discontinuous dynamics of stock price $S(t)$.

$$dS(t) = [aS(t) + \mu S(t-\tau)]\,dt + \sigma S(t-\rho)dW(t) + \int_{-1}^{\infty} yS(t)\nu(dt, dy) \qquad (5.3)$$

Dynamics of the bond price is a determined process satisfying an equation:

$$dB(t) = [bB(t) + \nu B(t-\beta)]dt \qquad (5.4)$$

Within coditions of (B, S)-market the following terms were considered.
Discounted stock price: $S^*(t) = S(t)/B(t)$, where $S(t)$ and $B(t)$ are defined by (5.3) and (5.4).
Capital of the portfolio holder: $X(t) = \alpha(t)S(t) + \beta(t)B(t)$, where: $(\alpha(t), \beta(t))$ is a portfolio at the time t consisting of $\alpha(t)$ stocks and $\beta(t)$ bonds.

The authors consider the following case studies of equations (5.3) and (5.4).

Theorem 25 *(1) Consider a stability problem for the following system.*

$$\begin{cases} dS(t) = [aS(t) + \mu S(t-\tau)]dt + \sigma S(t-\rho)dW(t) \\ dB(t) = [bB(t) + \nu B(t-\beta)]dt \end{cases} \qquad (5.5)$$

The following was obtained by constructing appropriate Lypunov functionals. Assume

$$a + \frac{\sigma^2}{2} + |\mu| < 0, \quad b + |\nu| < 0. \qquad (5.6)$$

Then the trivial solution of the system (5.5) *is exponentially stable.*

(2) Consider a discounted stock price $S^*(t) = S(t)/B(t)$ *with* $S(t)$ *and* $B(t)$ *satisfying the system of equations* (5.5). *Assume*

$$a + \frac{\sigma^2}{2} + |\mu| < 0. \qquad (5.7)$$

Then, a stochastic process $S^*(t)$ *(as a solution of appropriate equation) is exponentially stable.*

(3) Consider an equation for the stock price with jump component:

$$dS(t) = (aS(t) + \mu S(t-\tau))dt + \sigma S(t-\rho)dW(t) + \int_{-1}^{+\infty} yS(t)\nu(dt, dy) \qquad (5.8)$$

Assume

$$a + \frac{1}{2}\int_{-1}^{+\infty} y^2\Pi(dy) + |\mu| + \sigma^2/2 < 0 \tag{5.9}$$

Then, the trivial solution of (5.8) *is exponentially stable.*

(4) Let us consider the following capital process:

$$X(t) = \beta(t)B(t) + \gamma(t)S(t)$$

with $S(t)$ *and* $B(t)$ *defined by* (5.5). *Then,* $X(t)$*and* $Y(t) = -\beta(t)B(t) + \gamma(t)S(t)$ *satisfy the following system of equations (for simplicity we impose* $\beta = \rho = \tau = 1$*)*

$$dX_t = \frac{1}{2}\left[(a+b)X_t + (\nu\frac{\beta_t}{\beta_{t-1}} + \mu\frac{\gamma_t}{\gamma_{t-1}})X_{t-1} + (a-b)Y_t + (-\nu\frac{\beta_t}{\beta_{t-1}} + \mu\frac{\gamma_t}{\gamma_{t-1}})Y_{t-1}\right]dt$$
$$+\frac{1}{2}\sigma\frac{\gamma_t}{\gamma_{t-1}}(X_{t-1} + Y_{t-1})dW_t$$
$$dY_t = \frac{1}{2}[(a-b)X_t + (-\nu\frac{\beta_t}{\beta_{t-1}} + \mu\frac{\gamma_t}{\gamma_{t-1}})X_{t-1} + (a+b)Y_t + (\nu\frac{\beta_t}{\beta_{t-1}} + \mu\frac{\gamma_t}{\gamma_{t-1}})Y_{t-1}]dt$$
$$+\frac{1}{2}\sigma\frac{\gamma_t}{\gamma_{t-1}}(X_{t-1} + Y_{t-1})dW_t$$

The analysis of exponential stability of this system was reduced to the analysis of principal minors of the matrix 4×4.

5.3. In the article [126] (Kazmerchuk, Swishchuk, Wu, 2002) the authors consider an option pricing problem for financial securities market with delayed response. The following sdde is assumed to describe the evolution of stock price:

$$dS(t) = rS(t)dt + \sigma(t, S_t)S(t)dW(t) \tag{5.10}$$

with continuous deterministic initial data $S_0 = \varphi \in C := C([-\tau, 0], R)$, here σ represents a *volatility* which is a continuous function of time and the elements of C.

The existence and uniqueness of solution of (5.10) are guaranteed if the coefficients in (5.10) satisfy usual local Lipschitz and growth conditions. The authors were primarily interested in an option price value, which is assumed to depend on the current and previous stock price values in the following way:

$$F(t, S_t) = \int_{-\tau}^{0} e^{-r\theta}H(S(t+\theta), S(t), t)d\theta, \tag{5.11}$$

where $H \in C^{0,2,1}(R\times R\times R_+)$. Then, an analogue of Ito's formula for (5.10) and (5.11) was derived. Using this, the following general integral-differential equation was established.

Theorem 26 *Suppose the functional* F *is given by* (5.11) *with* $S(t)$ *satisfying* (5.10) *and* $H \in C^{0,2,1}(R \times R \times R_+)$. *Then,* $H(S(t+\theta), S(t), t)$ *satisfies the following equation*

$$0 = H|_{\theta=0} - e^{-r\theta}H|_{\theta=-\tau} + \int_{-\tau}^{0} e^{-r\theta}\left(H_3' + rS(t)H_2' + \frac{1}{2}\sigma^2(t, S_t)S^2(t)H_{22}''\right)d\theta \tag{5.12}$$

for all $t \in [0, T]$.

A simplifying assumptions was made, and the following closed-form solution of (5.12) was introduced:

$$F(t,S_t) = h_1(S(t),t) + \frac{1}{2}\int_t^T e^{r(t-\xi)}\left[\sigma^2(\xi,S_t) - V\right] S^2(t)\frac{\partial^2 h_1}{\partial S^2}(S(t),\xi)\ d\xi, \qquad (5.13)$$

where $h_1(S(t),t)$ is a classical Black-Scholes call option price with the variance assumed equal to a long-run variance rate V:

$$h_1(S(t),t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2), \qquad (5.14)$$

with $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-x^2}dx$ and d_1, d_2 defined as

$$d_1 = \frac{\ln(S(t)/K) + (r + V/2)(T-t)}{\sqrt{V(T-t)}},$$
$$d_2 = d_1 - \sqrt{V(T-t)}.$$

Also, authors derived a continuous-time analogue of discrete GARCH(1,1) model. The derived model appeared to be a stochastic volatility model with the variance rate $\sigma^2(t)$ satisfying

$$\frac{d\sigma^2(t)}{dt} = \gamma V + \frac{\alpha}{\tau}\ln^2\left(\frac{S(t)}{S(t-\tau)}\right) - (\alpha+\gamma)\sigma^2(t), \qquad (5.15)$$

where V is a long-run average variance rate, α and γ are positive constants such that $\alpha+\gamma<1$. Here, $S(t)$ is a solution of the sdde (5.10) with positive initial data $\varphi \in C$. A closed-form expression for $\sigma^2(t)$ was introduced which in combination with (5.13) led to the following formula for an option price:

$$F(t,S_t) = h_1(S(t),t) + (\Sigma(S_t)-V)\mathcal{I}(r,t,S(t)) + (\sigma^2(t)-\Sigma(S_t))\mathcal{I}(r+\alpha+\gamma,t,S(t)), \quad (5.16)$$

where $h_1(S(t),t)$ is given by (5.14) and

$$\Sigma(S_t) = \frac{\alpha}{\tau(\alpha+\gamma)}\ln^2\left(\frac{S(t)}{S(t-\tau)}\right) + \frac{\gamma V}{\alpha+\gamma},$$

$$\mathcal{I}(p,t,S(t)) = \frac{1}{2}S^2(t)\int_t^T e^{p(t-\xi)}\frac{\partial^2 h_1}{\partial S^2}(S(t),\xi)d\xi \quad for \quad p \geq 0.$$

5.4. In the article [110] (Chang, Youree, 1999) the authors propose a model for the (B,S)-market in which the dynamics of the stock price and the bank account are described by linear SDDEs. In particular, the pricing of the European contingent claims is studied and the corresponding trading strategy is derived.

Specifically, the authors assume that B and S evolve according to the following two linear stochastic functional differential equations:

$$dB(t) = L(B_t)dt,$$
$$dS(t) = M(S_t)dt + N(S_t)dW(t),$$

with initial price functions $B_0 = \phi$ and $S_0 = \psi$, where ϕ and ψ are given functions in $C_+ := \{\phi \in C[-h, 0] \mid \phi(\theta) \geq 0 \text{ for all } \theta \in [-h, 0]\}$. In the above, L, M and N are bounded linear functionals on the real Banach space $C[-h, 0]$.

It was established an existence of a unique probability measure $\tilde{P}$ such that

$$\tilde{P}(A) = E[\mathbf{1}_A Z(T)] \text{ for all } A \in \mathcal{F}_T,\ 0 < T < \infty,$$

where

$$Z(t) = \exp\left\{\int_0^t \gamma(B_s, S_s)dW(s) - \frac{1}{2}\int_0^t \gamma(B_s, S_s)^2 ds\right\},$$
$$\gamma(\phi, \psi) = \frac{\phi(0)M(\psi) - \psi(0)L(\phi)}{\phi(0)N(\psi)}.$$

Then, it was proved that the discounted wealth process

$$Y^\pi(t) = (\pi_1(t)B(t) + \pi_2(t)S(t))/B(t)$$

for a given self-financing trading strategy $(\pi_1(t), \pi_2(t))$ is a local martingale under the probability measure $\tilde{P}$.

The main results of this paper are summarized in the following theorem.

Theorem 27 *Let Λ be an $\mathcal{F}_T$-measurable random variable such that $E[\Lambda^{1+\varepsilon}] < \infty$ for some $\varepsilon > 0$. Then the rational price C of the contingent claim Λ is given by*

$$C = E_{\tilde{P}}[e^{-rT}\Lambda],$$

where r is a positive constant that satisfies $r = L(e^{r\cdot})$. Furthermore, there exists a minimal hedging strategy $\pi^* = \{(\pi_1^*(t), \pi_2^*(t)),\ 0 \leq t \leq T\}$.

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